Problem 1

Professor Chance Gardner is in a hurry to get to the class she is teaching. She can drive to campus either on I5 (denote this event by $I$) or the shorter way over the University Bridge (denote this event by $\bar{I}$). The latter can be either up (denote this event by $U$) or down. After she gets to campus she may find a parking spot (denote this event by $P$) or not (in which case she will have to park in the Montlake area). Finally, denote by $L$ the event “professor Gardner is late for class”. Below are given the probabilities of all possible individual outcomes of this random experiment.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$IPL$</td>
<td>0.05</td>
</tr>
<tr>
<td>$I\bar{P}L$</td>
<td>0.1</td>
</tr>
<tr>
<td>$I\bar{P}L$</td>
<td>0.15</td>
</tr>
<tr>
<td>$\bar{I}P\bar{L}$</td>
<td>0.18</td>
</tr>
<tr>
<td>$\bar{I}U\bar{P}L$</td>
<td>0.07</td>
</tr>
<tr>
<td>$\bar{I}UPL$</td>
<td>0.35</td>
</tr>
<tr>
<td>$\bar{I}UP\bar{L}$</td>
<td>0.1</td>
</tr>
</tbody>
</table>

The outcomes not listed (like $I\bar{P}\bar{L}=$"drives on I5, doesn’t find a parking space and is not late for class") have probability 0. Some “individual outcomes” in the above list are really events (for example $IPL$ which is the union of $IPLU$ and $IPL\bar{U}$). We still can call it an individual outcome assuming that if the route is over the I5 bridge then it doesn’t matter what is the state of University bridge (or we may not find out what this state is).

a. Make a neatly labeled drawing of this sample space, showing all the possible outcomes and their probabilities.

b. What is the probability that Chance drives over the University Bridge?

c. What is the probability that Chance makes it to class in time?

d. What is the probability that Chance doesn’t find a parking spot and is late
e. Which of the following events is more probable: $A$ = “Chance drives over the University bridge and is not late for her class,” or $B$ = “Chance drives over the I5 bridge and is not late for her class”?

**Problem 2**

a. Show that for any two events $A, B$, $P(A \cup B) \leq P(A) + P(B)$.

b. The ACME company manufactures the Pogo Thunder personal jet plane. The plane consists of three major systems: the body, the engines, and the electronics with failure probabilities of 0.01, 0.02 and 0.03 respectively (the Pogo Thunder is the least reliable personal jet on the market). The plane will work only if no system fails. Prove that the probability that the plane has no failure is larger or equal to 0.94.

c. (Optional, for extra credit) This problem is a special case of a very useful result in probability and measure theory called the *union bound*. Prove that for any events $A_1, A_2, \ldots, A_n \subseteq S$ the probability of the union is no larger than the sum of the events’ probabilities, i.e.

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) \leq P(A_1) + P(A_2) + \ldots + P(A_n) \quad (1)$$

**Problem 3**

Let $S = \{0, 1, 2, \ldots 9\}$ be the sample space of the 10 digits and $P(n)$ be an exponential distribution over this space, given by

$$P(n) = \frac{1}{Z} \gamma^n \quad (2)$$

with $\gamma = 1/2$.

a. What is the value of the normalization constant $Z$? Give either a numerical answer or a formula in terms of $\gamma$.

b. What is the probability that $n < 5$? Give either a numerical answer or a formula in terms of $\gamma$.

c. What is the probability that $n$ is odd? Give either a numerical answer or a formula in terms of $\gamma$.

d. Let $S_3 = S^3$ be the sample space of all sequences of 3 elements from $S$. How many elements has $S_3$?

e. We assume that the sequences in $S_3$ are obtained by sampling independently
3 times from $S$ with the probability distribution $P$ defined in (2). Compute the probability of the sequences $(1, 2, 3)$, $(0, 0, 0)$ and $(0, 1, 1)$.

**f.** What is the sequence $(n^{(1)}, n^{(2)}, n^{(3)})$ that has highest probability of occurrence? What is the sequence with the lowest probability?

**g.** For any outcome $(n^{(1)}, n^{(2)}, n^{(3)})$ define

$$z(n^{(1)}, n^{(2)}, n^{(3)}) = 100n^{(1)} + 10n^{(2)} + n^{(3)},$$

i.e $z(n^{(1)}, n^{(2)}, n^{(3)})$ is the decimal number whose digits are drawn randomly and independently from $P$. For instance $z(2, 0, 6) = 206$ and $z(0, 1, 2) = 12$.

Let $(n^{(1)}, n^{(2)}, n^{(3)})$ and $(\tilde{n}^{(1)}, \tilde{n}^{(2)}, \tilde{n}^{(3)})$ be two sequences with $z(n^{(1)}, n^{(2)}, n^{(3)}) > z(\tilde{n}^{(1)}, \tilde{n}^{(2)}, \tilde{n}^{(3)})$. Is it true that this implies

$$P(n^{(1)}, n^{(2)}, n^{(3)}) \leq P(\tilde{n}^{(1)}, \tilde{n}^{(2)}, \tilde{n}^{(3)}) \quad (3)$$

In other words, is it true that the resulting probability distribution over 3 digit integers is monotonically decreasing?

Prove or give a counterexample.

FYI: The sum of the geometric progression

$$a + ax + ax^2 + ax^3 + \ldots + ax^{m-1} = a\frac{1-x^m}{1-x}$$