STAT 391
Final Exam
10:30 – 12:20 on June 9, 2005
©Marina Meilä
mmp@cs.washington.edu

Student Name: _______________________________
Prob.1 ___ of 6
Prob.2 ___ of 5
Prob.3 ___ of 6
Prob.4 ___ of 5
Prob.5 ___ of 4
Prob.6 ___ of 9
Bonus 1 TOTAL: ____ of 36

You are allowed 10 pages of notes and the course pack if it is bounded. Please
print your name on top of each of the pages of notes and on the course pack.
Calculators and other electronic devices are not allowed.

Any fact that was proved in the lectures or in the notes can be used without proof.

Do well!
Problem 1

The following figures represent 500 pairs $X,Y$ whose joint density $f_{XY}$ is a two-dimensional normal distribution. Check all boxes for which the following statements are true:

1.1. $X$ and $Y$ are independent

1.2. $\text{Cov}(X,Y) < 0$

1.3. $\text{Var} X > \text{Var} Y$

1.4. $E[Y] > 0$

A  B  C  D  E  F

A  B  C  D  E  F

A  B  C  D  E  F

A  B  C  D  E  F
Problem 2

A fair coin is tossed 2 times; denote by $X_i \in S = \{0,1\}$ the outcome of toss $i$, $i = 1,2$. Assume that the probability of getting an outcome of 1 on either toss is $p = 0.5$ and that the coin tosses are independent events.

1 p.  2.1 What is the variance of $X_1$?

1 p.  2.2 On toss $i$, you gain $i$ dollars if the outcome is 1 and nothing if the outcome is 0. Let the random variable $Y$ represent the total gain from the two tosses. Express $Y$ as a function of $X_1$ and $X_2$.

1 p.  2.3 What is the probability distribution of $Y$?
1 p.  2.4 What is the variance $\text{Var} Y$?

1 p.  2.5 What is the conditional probability that $X_2 = 1$ given that $Y > 0$?
6 points

Problem 3

The following graphs represent unnormalized probability densities over the real axis.

1 p. 3.1 Compute the normalization constants for the two densities; write their values above the respective graphs.

2 p. 3.2 Draw the CDF’s of the two densities in the corresponding plots.

2 p. 3.3 Mark the position of the mean $\mu = E[X]$ on each $f$ graph.

1 p. 3.4 What is the probability of the interval $[-1, 2]$ under each distribution? (Just the numerical value OK; no derivation required)
Problem 4

Random variables $X$ and $Y$ are normally distributed and independent with means $\mu_x = \mu_y = 0$ and variances $\sigma_x^2, \sigma_y^2$. Denote

\[
U = X + Y \\
V = X - Y
\]

1 p. 4.1 What are the expectations $\mu_u = E[U]$ and $\mu_v = E[V]$?

1 p. 4.2 What are the variances $\sigma_u^2 = Var(U)$ and $\sigma_v^2 = Var(V)$?
1 p. **4.3** Compute the expression of the covariance $\text{Cov}(U, V)$.

1 p. **4.4** Could $U, V$ be independent? Under what conditions?

1 p. **4.5** Assuming that $\sigma_x = 1$ and $\sigma_y = 2$, compute the mean and covariance matrix of the joint probability density of $U, V$. 
4 points **Problem 5**

Compute and make neatly labeled graphs of the density estimate for each of the (very small) data sets given below. If you choose to draw an unnormalized f write the value of the normalization constant next to the plot.

**Example Q:** Find an exponential density estimate; data = \{1, 1, 2\}

**Example A:** $f_\lambda(x) = \lambda e^{-\lambda x}$ with $\lambda = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{3}{1+1+2} = \frac{3}{4}$

5.1 Find a kernel density estimate, for a square kernel with $h = 2$. Assume that the square kernel is $k(x) = \begin{cases} 1 & \text{if } x \in [-0.5, 0.5] \\ 0 & \text{otherwise} \end{cases}$ data = \{0, 1, 3\}
5.2 Find a normal density estimate.

data = \{-1, 0, 1, 4\} (A table with square roots of numbers 2–20 is available at the bottom of the page.)

<table>
<thead>
<tr>
<th>$z$</th>
<th>$\sqrt{z}$</th>
<th>$z$</th>
<th>$\sqrt{z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>1.41</td>
<td>12.0</td>
<td>3.46</td>
</tr>
<tr>
<td>2.5</td>
<td>1.58</td>
<td>12.5</td>
<td>3.54</td>
</tr>
<tr>
<td>3.0</td>
<td>1.73</td>
<td>13.0</td>
<td>3.61</td>
</tr>
<tr>
<td>3.5</td>
<td>1.87</td>
<td>13.5</td>
<td>3.67</td>
</tr>
<tr>
<td>4.0</td>
<td>2.00</td>
<td>14.0</td>
<td>3.74</td>
</tr>
<tr>
<td>4.5</td>
<td>2.12</td>
<td>14.5</td>
<td>3.81</td>
</tr>
<tr>
<td>5.0</td>
<td>2.24</td>
<td>15.0</td>
<td>3.87</td>
</tr>
<tr>
<td>5.5</td>
<td>2.35</td>
<td>15.5</td>
<td>3.94</td>
</tr>
<tr>
<td>6.0</td>
<td>2.45</td>
<td>16.0</td>
<td>4.00</td>
</tr>
<tr>
<td>6.5</td>
<td>2.55</td>
<td>16.5</td>
<td>4.06</td>
</tr>
<tr>
<td>7.0</td>
<td>2.65</td>
<td>17.0</td>
<td>4.12</td>
</tr>
<tr>
<td>7.5</td>
<td>2.74</td>
<td>17.5</td>
<td>4.18</td>
</tr>
<tr>
<td>8.0</td>
<td>2.83</td>
<td>18.0</td>
<td>4.24</td>
</tr>
<tr>
<td>8.5</td>
<td>2.92</td>
<td>18.5</td>
<td>4.30</td>
</tr>
<tr>
<td>9.0</td>
<td>3.00</td>
<td>19.0</td>
<td>4.36</td>
</tr>
<tr>
<td>9.5</td>
<td>3.08</td>
<td>19.5</td>
<td>4.42</td>
</tr>
<tr>
<td>10.0</td>
<td>3.16</td>
<td>20.0</td>
<td>4.47</td>
</tr>
<tr>
<td>10.5</td>
<td>3.24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11.0</td>
<td>3.32</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11.5</td>
<td>3.39</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
9 points  **Problem 6**

The new CSE building has an intelligent elevator, named Ella, who carries visitors to the 6 floors of the building. In her idle time, Ella exercises her mind with probability questions. Can you help her find the answers?

Assume in the following that even if a person goes to floor 1, they will still stop by the elevator.

1 p.  **6.1** Ella has noticed that in a given time interval, there were 10 visitors which requested the following floors:

\[ 6 \ 1 \ 2 \ 3 \ 5 \ 4 \ 5 \ 6 \ 6 \ 5 \]

Let \( F \) denote the random variable designating the floor a person will go to. Assuming that the visitors are all independent of each other, estimate from the data above the parameters \( \theta_1, \theta_2, \ldots, \theta_6 \) of the probability distribution of \( F \), where \( \theta_i = P_F(i) \).

Use the estimates obtained in **6.1** to answer the next questions.

1 p.  **6.2** What is the probability that a person goes to an even numbered floor?
6.3 In front of the elevator are 4 persons. What is the probability that at least one of them is going to an even numbered floor?

6.4 The Artificial Intelligence (AI), Graphics (G), Systems (S) and Theory (T) groups are located according to the “map” below. In other words, AI groups are on the 3-rd and 4-th floors, graphics on 4 and 6, etc. Assume that there are no other groups on floors 3,4 and 6, that a visitor to floor i will be going to only one group, and that groups on the same floor have equal probabilities of being visited. E.g a visitor to floor 6 will go to graphics w.p 0.5 and to systems w.p 0.5.

<table>
<thead>
<tr>
<th>6</th>
<th>G</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>AI</td>
<td>G</td>
</tr>
<tr>
<td>4</td>
<td>G</td>
<td>S</td>
</tr>
<tr>
<td>3</td>
<td>AI</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What is the probability that a person goes to the graphics group?
6.5 What is the probability that a person goes to the 6th floor, given that they are headed for the graphics group?

6.6 There are 2 people in the elevator. What is the probability that they go to the same floor, given that they are both going to the graphics group?
6.7 Today is a holiday, but it is also a deadline for a major AI conference. Therefore, a person going to the G, S or T groups will stay more than an hour (call this time “long”) with probability \(p\), but a person going to the AI group will stay long with probability \(q\), \(q > p\).

Alice and Bob went to the same floor (call this event \(E_1\)). Alice stayed long (call this event \(L_A\)) and Bob didn’t go to the Theory group (call this event \(E_2\)). Under these conditions, what is the probability that Bob stayed long (call this event \(L_B\))? 

\[ \text{That’s all folks!} \]