Lecture 8: Limits and Time to Extinction of the Endemic Process, and a touch of Quasi Stationary Processes, Yaglom’s Limit, Chaos
**Expected Length of the Epidemic (Endemic Model)\(^*\)**

\((q = .95)\)

<table>
<thead>
<tr>
<th>Population Size</th>
<th>Expected Length of Epidemic</th>
<th>(i = 1)</th>
<th>(i = 2)</th>
<th>(i = 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>1.23</td>
<td>1.33</td>
<td>1.37</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>1.63</td>
<td>1.98</td>
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<tr>
<td>15</td>
<td></td>
<td>2.30</td>
<td>3.02</td>
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<tr>
<td>20</td>
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<td>3.76</td>
<td>5.21</td>
<td>6.04</td>
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<tr>
<td>25</td>
<td></td>
<td>8.67</td>
<td>12.40</td>
<td>14.39</td>
</tr>
<tr>
<td>30</td>
<td></td>
<td>45.21</td>
<td>62.89</td>
<td>70.78</td>
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<td>35</td>
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<td>1,160.22</td>
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<td>40</td>
<td></td>
<td>52,385.00</td>
<td>63,022.00</td>
<td>65,452.00</td>
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</tbody>
</table>

*Source: Longini, I.M. A Chain Binomial Model of Endemicity, Mathematical Biosciences 50: 85-93 (1980).*
Deterministic System

\[ I_{t+1} = (n - I_t)(1 - p \cdot I_t) \]

a) If \(-n \ln q \leq 2\), then \(\lim_{t \to \infty} I_t = 0\)

b) If \(-n \ln q > 2\), then \(\lim_{t \to \infty} I_t = I^* > 0\),

where \(I^*\) is root of \(I^* = (n - I^*)(1 - p I^*)\)

Note: For \(p = 1 - q \approx 0\), \(-n \ln q \approx np = R_0\)

E.g., \(q = 0.95\), \(-\ln 0.95 = 0.051\)

\(n = 19\), \(-19 \ln 0.95 = 0.97\) \(I^* = 0\)

\(n = 20\), \(-20 \ln 0.95 = 1.03\) \(I^* = 1^+\)

\(n = 30\), \(-30 \ln 0.95 = 1.54\) \(I^* = 6.9\)

\(n = 50\), \(-50 \ln 0.95 = 2.56\) \(I^* = 19.3\)
\[ x_t = \frac{I_t}{n} \quad \varphi = e^{-\beta/n} = e^{-\alpha} \]

\[ x_{t+1} = (1-x_t)(1-e^{-\alpha x_t}) \]

\[ 1 - e^{-\alpha x_t} \approx \alpha x_t \quad \text{(First Order Approx.)} \]

\[ x_{t+1} = (1-x_t)\alpha x_t = \alpha x_t - \alpha x_t^2 \quad \text{(Discrete Logistic Equation)} \]

\[ 0 \leq \alpha \leq 1 \quad x^* = 0 \]

\[ 1 < \alpha < 3^- \quad x^* > 0 \quad \text{(Constant)} \]

\[ \alpha = 3 \quad \text{Stable 2-Cycle} \]

\[ \alpha > 3 \quad \text{Cycles of 4, 8, 16} \]

\[ \alpha >> 3 \quad \text{Infinitely Many Cycles "Chaos"} \]


SIMULATION WHEN $N = 30$, $c_0 = 1$ AND $\gamma = 0.45$ * ($m_1 = 45.21$)

STOCHASTIC ..........
DETERMINISTIC ____

\[
\mu = \frac{1}{30} \sum_{t=1}^{30} I_t = 6.08
\]

DETERMINISTIC SYSTEM: \(-N \ln \gamma = -30 \ln 0.55 = 1.534\)

Solution of \(I = (N-I)(1-\gamma^I) = (30-I)(1-0.55^I)\)

IS \(I = 6.187\)

**Conditioning on Non-Extinction**

Let \( q_{i,j}(t) = P[I(t) = j \mid I(t) \neq 0, I(0) = i] \)

\[ q_{j,k} = \frac{p_{j,k}}{1 - p_{j,0}} \text{ conditioned one-step trans. prob.} \]

\[ q_{i,j}(t+1) = \sum_{k=1}^{N-1} q_{i,k}(t) q_{k,j} \quad \text{and} \]

\[ \lim_{t \to \infty} q_{i,j}(t) = \pi_j, \quad j = 1, \ldots, N-1 \quad \text{Yaglom's limit} \]

\[ \pi_j = \sum_{k=1}^{N-1} \pi_k q_{k,j}, \quad j = 1, \ldots, N-1 \]

\[ \sum_{j=1}^{N-1} \pi_j = 1 \]

**Quasi Stationary Process**
**Conditional Absorption**

**Not Absorbed**

\[ a(t+1) = P[x(t+1) \neq 0 \mid x(t) \neq 0] \]

**Absorbed**

\[ 1 - a(t+1) = \sum_{k=1}^{N-1} q_{ik}(t) q^{(N-k)k} \]

\[ \lim_{t \to \infty} a(t) = a \]

\[ 1 - a = \prod_{k=1}^{N-1} q^{(N-k)k} \]
<table>
<thead>
<tr>
<th>$q$</th>
<th>$\pi_1$</th>
<th>$\pi_2$</th>
<th>$\pi_3$</th>
<th>$\pi_4$</th>
<th>$\pi_5$</th>
<th>$\text{ECT}/I_0 = 1$</th>
<th>$1-a$</th>
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<tbody>
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