Stochastic Processes

- "bowl of worms" (Figure 31)
- stochastic process is:
  - informally: bowl + drawing mechanism
  - formally: family of random variables (rvs) indexed by t
  - t called "time" for convenience
- real-valued rv: mapping from sample space to real line
- denote individual real-valued rvs by X_t or $X(t)$
- notation:
  - *discrete* parameter stochastic process:

$$\{X_t\} = \{X_t : t = 0, \pm 1, \pm 2, \ldots\}$$

  - *continuous* parameter stochastic process:

$$\{X(t)\} = \{X(t) : -\infty < t < \infty\}$$

Basic Theory for Stochastic Processes

- cumulative probability distribution function (cpdf):

$$F_t(a) \equiv \mathbf{P}[X_t \le a]$$

- can define bivariate cpdf (and multivariate cpdf):

$$F_{t_1,t_2}(a_1, a_2) \equiv \mathbf{P}[X_{t_1} \le a_1, X_{t_2} \le a_2]$$

- note dependence on t
  - too complicated to use as models for time series
  - summarize using moments (assuming they exist)
- first moment:

$$\begin{aligned} \mu_t \equiv E\{X_t\} &= \int_{-\infty}^{\infty} x \, dF_t(x) \quad \text{(see page 34)} \\ &= \int_{-\infty}^{\infty} x f_t(x) \, dx \quad \text{if } f_t(x) = \frac{dF_t(x)}{dx} \text{ exists} \\ &= \sum_i x_i p(x_i) \quad \text{if } F_t(\cdot) \text{ is a step function} \end{aligned}$$

- second-order central moments:

$$\begin{aligned} \sigma_{t_1,t_2} &\equiv \text{cov}\{X_{t_1}, X_{t_2}\} = E\{(X_{t_1} - \mu_{t_1})(X_{t_2} - \mu_{t_2})\} \\ \sigma_t^2 &\equiv \text{var}\{X_t\} = \int_{-\infty}^{\infty} (x - \mu_t)^2 \, dF_t(x) \end{aligned}$$

- still too complicated!

Simplification: Stationary Processes

- $\{X_t\}$ is by definition stationary if:
  1. $E\{X_t\} = \mu$, a finite constant independent of t
  2. $\text{cov}\{X_t, X_{t+\tau}\} = s_\tau$, a finite constant that can depend on τ, but not t
- τ is called the *lag*
- autocovariance sequence (acvs) for $\{X_t\}$:

$$\{s_\tau\} = \{s_\tau : \tau = 0, \pm 1, \pm 2, \ldots\}$$

- autocovariance function (acvf) for $\{X(t)\}$:

  $s(\cdot) =$ function of τ defined by $s(\tau)$ over $-\infty < \tau < \infty$
- terms for this type of stationarity:
  - second-order
  - covariance
  - wide sense
  - weak
- other type of stationarity: strict stationarity

Basic Properties: I

- $\text{var}\{X_t\} = \text{cov}\{X_t, X_t\} = s_0$
  (i.e., variance constant over time)
- $s_{-\tau} = s_\tau$ (i.e., symmetric about $\tau = 0$)
- correlation between X_t and $X_{t+\tau}$:

$$\begin{aligned}\text{corr}\{X_t, X_{t+\tau}\} &\equiv \frac{\text{cov}\{X_t, X_{t+\tau}\}}{\sqrt{\text{var}\{X_t\}\,\text{var}\{X_{t+\tau}\}}} \\ &= \frac{\text{cov}\{X_t, X_{t+\tau}\}}{\text{var}\{X_t\}} = \frac{s_\tau}{s_0} \equiv \rho_\tau\end{aligned}$$

  – autocorrelation sequence (acs) for $\{X_t\}$:

$$\{\rho_\tau\} = \{\rho_\tau : \tau = 0, \pm 1, \pm 2, \ldots\}$$

  – autocorrelation function (acf) for $\{X(t)\}$:

  $\rho(\cdot)$ = function defined by $\rho(\tau)$ over $-\infty < \tau < \infty$

  – $|\rho_\tau| \leq 1$ implies $|s_\tau| \leq s_0$

Basic Properties: II

- $\{s_\tau\}$ is positive semidefinite, i.e.,

$$\sum_{j=1}^{n}\sum_{k=1}^{n} s_{t_j - t_k} a_j a_k \geq 0 \qquad (1)$$

  holds for

  - any postive integer n
  - any set of n real numbers $a_1, \ldots, a_n$
  - any set of n integers $t_1, \ldots, t_n$

- proof: for $W \equiv \sum_{j=1}^{n} a_j X_{t_j} = \mathbf{a}^T \mathbf{V}$, have

$$0 \leq \text{var}\{W\} = \text{var}\{\mathbf{a}^T \mathbf{V}\} \stackrel{[2.8]}{=} \mathbf{a}^T \Sigma \mathbf{a}$$

  - Σ is var/cov matrix for $\mathbf{V}$
  - (j,k)th element is $\text{cov}\{X_{t_j}, X_{t_k}\} = s_{t_j - t_k}$
  - $\mathbf{a}^T \Sigma \mathbf{a} = (1)$

- imposes severe limitation (Makhoul, 1990)

Basic Properties: III

- var/cov matrix for $[X_1,\ X_2,\ \ldots,\ X_N]^T$ is Toeplitz
  - e.g., $N = 7$:

$$\begin{bmatrix} s_0 & \underline{s_1} & s_2 & s_3 & s_4 & s_5 & s_6 \\ s_1 & s_0 & \underline{s_1} & s_2 & s_3 & s_4 & s_5 \\ s_2 & s_1 & s_0 & \underline{s_1} & s_2 & s_3 & s_4 \\ s_3 & s_2 & s_1 & s_0 & \underline{s_1} & s_2 & s_3 \\ s_4 & s_3 & s_2 & s_1 & s_0 & \underline{s_1} & s_2 \\ s_5 & s_4 & s_3 & s_2 & s_1 & s_0 & \underline{s_1} \\ s_6 & s_5 & s_4 & s_3 & s_2 & s_1 & s_0 \end{bmatrix}$$

- if $\{X_t\}$ Gaussian, completely characterized by
  - μ
  - $\{s_\tau\}$
- extension: complex-valued stationary process $\{Z_t\}$
  - $\text{cov}\{Z_t, Z_{t+\tau}\} \equiv E\{(Z_t - \mu)^*(Z_{t+\tau} - \mu)\} = s_\tau$ (asterisk = complex conjugation)
  - now have $s_{-\tau} = s_\tau^*$ (rather than $s_{-\tau} = s_\tau$)

Example: White Noise Process

- assume $E\{X_t\} = \mu$ and $\text{var}\{X_t\} = \sigma^2$ $(< \infty)$
- $\{X_t\}$ called white noise if, when $\tau \neq 0$,
  X_t & $X_{t+\tau}$ uncorrelated; i.e., $\text{cov}\{X_t, X_{t+\tau}\} = 0$
- white noise is stationary process with acvs

$$s_\tau = \begin{cases} \sigma^2, & \tau = 0; \\ 0, & \text{otherwise.} \end{cases}$$

- examples of realizations: Figures 41 and 42
  (note: all have *same* second-order properties!)
- note: distributions of X_t and X_{t+1} can be different!
- related concept: IID noise
  - $\{X_t : t = 0, \pm 1, \pm 2, \ldots\}$ are independent & identically distributed
  - all IID noise with finite variance is white noise; converse is false

Example: Moving Average Process

- let $\{\epsilon_t\}$ be white noise, mean 0, variance σ_ϵ^2
- construct MA(q) process

$$X_t = \mu - \sum_{j=0}^{q} \theta_{j,q}\epsilon_{t-j} = \mu - \sum_{j=-\infty}^{\infty} \theta_{j,q}\epsilon_{t-j}$$

  with $\theta_{j,q} = 0$ for $j < 0$ or $j > q$; $\theta_{0,q} \neq 0$; $\theta_{q,q} \neq 0$;

- claim: $\{X_t\}$ is stationary
- proof: need to show
  - $E\{X_t\} = \mu$ (why true?)
  - cov$\{X_t, X_{t+\tau}\}$ can be expressed as s_τ:

$$\begin{aligned} \text{cov}\{X_t, X_{t+\tau}\} &= E\left\{\left(-\sum_j \theta_{j,q}\epsilon_{t-j}\right)\left(-\sum_{j'} \theta_{j',q}\epsilon_{t+\tau-j'}\right)\right\} \\ &= \sum_j \sum_{j'} \theta_{j,q}\theta_{j',q}E\{\epsilon_{t-j}\epsilon_{t+\tau-j'}\} \\ &= \sum_{j=0}^{q} \theta_{j,q}\theta_{j+\tau,q}\sigma_\epsilon^2 \qquad \text{(why?)} \\ &\equiv s_\tau \end{aligned}$$

- note: no restrictions on $\theta_{j,q}$'s
- example of MA(1) process: $X_t = \epsilon_t - \theta_{1,1}\epsilon_{t-1}$
  - Figure 44: $\theta_{1,1} = \pm 1$

Example: Autoregressive Process

- let $\{\epsilon_t\}$ be white noise, mean 0, variance σ_ϵ^2
- construct AR(p) process

$$X_t = \mu + \sum_{j=1}^{p} \phi_{j,p}(X_{t-j} - \mu) + \epsilon_t$$

  with $\phi_{p,p} \neq 0$

- claim: $\{X_t\}$ is stationary under restrictions on $\phi_{j,p}$'s
- proof: Priestley (1981)
- rich class of processes (Chapter 9)
- example of AR(2) process (Figure 45, top):

$$X_{t,2} = 0.75X_{t-1,2} - 0.5X_{t-2,2} + \epsilon_{t,2}$$

- example of AR(4) process (Figure 45, bottom):

$$\begin{aligned} X_{t,4} &= 2.7607X_{t-1,4} - 3.8106X_{t-2,4} \\ &\quad +2.6535X_{t-3,4} - 0.9238X_{t-4,4} + \epsilon_{t,4} \end{aligned}$$

- extension: ARMA(p,q) process (Box & Jenkins, 1970)

Example: Harmonic Process – I

- for fixed $L > 0$, let $\phi_1, \ldots, \phi_L$ be independent rv's uniformily distributed over $[-\pi, \pi]$
- construct harmonic process:

$$X_t = \mu + \sum_{l=1}^{L} D_l \cos(2\pi f_l t + \phi_l),$$

  where μ, D_l and f_l are real-valued constants
- what do realizations of $\{X_t\}$ look like?
- claim: $\{X_t\}$ is stationary (!)

Example: Harmonic Process – II

- proof when $L = 1$, i.e., $X_t = \mu + D_1 \cos(2\pi f_1 t + \phi_1)$
- need to show
  1. $E\{X_t\}$ is a constant:

$$\begin{aligned} E\{X_t\} &= \mu + D_1 E\{\cos(2\pi f_1 t + \phi_1)\} \\ &= \mu + D_1 \int_{-\pi}^{\pi} \cos(2\pi f_1 t + \phi_1) \frac{1}{2\pi}\, d\phi_1 \quad \text{(why?)} \\ &= \mu \end{aligned}$$

  2. $\text{cov}\{X_t, X_{t+\tau}\}$ can be expressed as s_τ:

$$\begin{aligned} &\text{cov}\{X_t, X_{t+\tau}\} \\ &\quad = E\left\{(D_1 \cos(2\pi f_1 t + \phi_1))\,(D_1 \cos(2\pi f_1 [t+\tau] + \phi_1))\right\} \\ &\quad = D_1^2 \int_{-\pi}^{\pi} \cos(2\pi f_1 t + \phi_1) \cos(2\pi f_1 [t+\tau] + \phi_1) \frac{1}{2\pi}\, d\phi_1 \\ &\quad \overset{\text{trig}}{=} \frac{D_1^2}{4\pi} \int_{-\pi}^{\pi} \cos(2\pi f_1 [2t+\tau] + 2\phi_1) + \cos(2\pi f_1 \tau)\, d\phi_1 \\ &\quad \equiv D_1^2 \cos(2\pi f_1 \tau)/2 \equiv s_\tau \end{aligned}$$

- basis for spectral representation theorem (Chapter 4)
- related (more general) model: random amplitudes

Stationary Processes as Models

- stationarity property of models, *not* data
- spectral analysis assumes stationarity
- need to examine assumption for each time series
  - example: spinning rotor series (Figure 50)
    - detrend using least squares model

$$X_t = \alpha + \beta t + Y_t$$

    - detrend using first difference (filtering)

$$X_t^{(1)} \equiv X_t - X_{t-1} = \beta + Y_t^{(1)}, \text{ where } Y_t^{(1)} \equiv Y_t - Y_{t-1}$$

  - example: standard resistor series (Figure 52)