Fourier Theory: Overview

- spectral analysis = stationary processes + Fourier
- basic idea behind Fourier theory: given
  - real/complex-valued function $g(\cdot)$ over $(-\infty, \infty)$
    or
  - real/complex-valued sequence $\{g_t : t = 0, \pm 1, \pm 2, \ldots\}$

  want to write (represent, synthesize) $g(\cdot)$ or $\{g_t\}$ as

  $$\text{“}\sum_{f \geq 0}\text{”} A(f) \cos(2\pi f t) + B(f) \sin(2\pi f t) = \text{“}\sum_{f}\text{”} C(f) e^{-i2\pi f t}$$

  where $e^{ix} \equiv \cos(x) + i \sin(x)$ and $i = \sqrt{-1}$

Four Flavors of Fourier Theory

- $g(\cdot)$ periodic with period T
  - "$\sum_f$" is sum over $f_n = n/T$, $n = 0, \pm 1, \pm 2, \ldots$
  - continuous time/discrete frequency
- $g(\cdot)$ square integrable: $\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$
  - "$\sum_f$" is integral over $(-\infty, \infty)$
  - continuous time/continuous frequency
- $\{g_t\}$ square summable: $\sum_{t=-\infty}^{\infty} |g_t|^2 < \infty$
  - "$\sum_f$" is integral over $[-1/2, 1/2]$
  - discrete time/continuous frequency
- $\{g_t : t = 0, 1, \ldots, N-1\}$, a finite sequence
  - "$\sum_f$" is sum over $f_n = n/N$, $n = 0, 1, \ldots, N-1$
  - discrete time/discrete frequency
- all used in spectral analysis!
- task: define $C(f)$ for each flavor (known as Fourier coefficients)

Cont. Time/Disc. Frequency: I

- assumptions
  - $g_p(\cdot)$ periodic with period T: $g_p(t+T) = g_p(t)$
  - $g_p(\cdot)$ square integrable over one period:

$$\int_{-T/2}^{T/2} |g_p(t)|^2 \, dt < \infty$$

- definitions
  - nth Fourier coefficient, $n = 0, \pm 1, \pm 2, \ldots$:

$$G_n \equiv \frac{1}{T} \int_{-T/2}^{T/2} g_p(t) e^{-i2\pi f_n t} \, dt, \quad f_n \equiv \frac{n}{T}$$

    interpretation of G_n: covariance between $g_p(\cdot)$ and complex exponential (if similar, $|G_n|$ large)
  - mth order Fourier approximation:

$$g_{p,m}(t) \equiv \sum_{n=-m}^{m} G_n e^{i2\pi f_n t}$$

    (least squares approximation – see pp. 60–2)

Cont. Time/Disc. Frequency: II

- can show:

$$\lim_{m\to\infty} \int_{-T/2}^{T/2} |g_p(t) - g_{p,m}(t)|^2 \, dt = 0$$

- shorthand for above:

$$g_p(t) \overset{\text{ms}}{=} \sum_{n=-\infty}^{\infty} G_n e^{i2\pi f_n t}$$

  RHS is Fourier series representation of $g_p(\cdot)$
  (ms equality is *not* pointwise equality – see pp. 62–3)

- notation:

$$g_p(\cdot) \longleftrightarrow \{G_n\}$$

- Parseval's theorem:

$$\int_{-T/2}^{T/2} |g_p(t)|^2 \, dt = T \sum_{n=-\infty}^{\infty} |G_n|^2$$

  LHS is "energy" in $g_p(\cdot)$ over $[-T/2, T/2]$

- corollary (why?):

$$\int_{-T/2}^{T/2} |g_p(t) - g_{p,m}(t)|^2 \, dt = T \sum_{|n|>m} |G_n|^2$$

Cont. Time/Disc. Frequency: III

- corollary:
$$\frac{1}{T}\int_{-T/2}^{T/2} |g_p(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |G_n|^2$$
LHS is “power” in $g_p(\cdot)$ (related to variance)
- what are energy & power over $[-mT/2, mT/2]$?
- can decompose power into pieces associated with f_n’s
- define discrete power spectrum for $g_p(\cdot)$: $S_n \equiv |G_n|^2$
- can we recover $g_p(\cdot)$ from S_n?
- first example (Figures 61–2)
  - 2π periodic function of Equation (60a):
  $$g_p(t) \equiv \frac{1-\phi^2}{1+\phi^2-2\phi\cos(t)}.$$
  (related to AR(1) process)
  - if $|\phi| < 1$, square integrable & $G_n = \phi^{|n|}$
  - mth order Fourier approximation:
  $$g_{p,m}(t) = \sum_{n=-m}^{m} G_n e^{i2\pi f_n t} = 1 + 2\sum_{n=1}^{m} \phi^n \cos(nt)$$
- two other examples: homework exercise

Cont. Time/Cont. Frequency: I

- assumption
  - $g(\cdot)$ square integrable: $\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$
- definition
  - Fourier transform (or analysis) of $g(\cdot)$:

$$G(f) \equiv \int_{-\infty}^{\infty} g(t) e^{-i2\pi f t} \, dt, \qquad -\infty < f < \infty$$

- can recover $g(\cdot)$ from $G(\cdot)$ (Fourier synthesis):

$$g(t) \overset{\text{ms}}{=} \int_{-\infty}^{\infty} G(f) e^{i2\pi f t} \, df$$

  $g(\cdot)$ is inverse Fourier transform of $G(\cdot)$

- motivate above using $g_{p,m}(\cdot)$ (Figure 64; pp. 64–5):

$$\begin{aligned} g(t) \equiv g_T(t) &\overset{\text{ms}}{=} \sum_{n=-\infty}^{\infty} \left(\int_{-T/2}^{T/2} g(u) e^{-i2\pi f_n u} \, du \right) e^{i2\pi f_n t} \frac{1}{T} \\ &\approx \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(u) e^{-i2\pi f u} \, du \right) e^{i2\pi f t} \, df \end{aligned}$$

Cont. Time/Cont. Frequency: II

- shorthand for above:

$$g(\cdot) \longleftrightarrow G(\cdot)$$

  - $G(\cdot)$ is Fourier transform of $g(\cdot)$
  - $g(\cdot)$ is inverse Fourier transform of $G(\cdot)$
  - $g(\cdot)$ & $G(\cdot)$ form a Fourier transform pair
  - many different conventions!

- Parseval's theorem:

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} |G(f)|^2 \, df$$

  LHS is "energy" in $g(\cdot)$ over $(-\infty, \infty)$
  (what would "power" be?)

- define energy spectral density function: $|G(f)|^2$
- can write $G(f) = |G(f)|e^{i\theta(f)}$
  - $|G(f)|$ defines amplitude spectrum
  - $\theta(f)$ defines phase function

Cont. Time/Cont. Frequency: III

- example (Figure 67):

$$e^{-\pi t^2} \longleftrightarrow e^{-\pi f^2}$$

  - change of variable yields $g_\sigma(\cdot) \longleftrightarrow G_\sigma(\cdot)$, where

$$g_\sigma(t) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-t^2/(2\sigma^2)}, \qquad G_\sigma(f) = e^{-2\pi^2 f^2 \sigma^2}$$

  - note: $G_\sigma(\cdot)$ real-valued so can see $|G_\sigma(\cdot)|^2$ easily

Time-/Band-Limited Functions

- $g(\cdot)$ time-limited to $[-T, T]$ if:
  $g(t) = 0$ for all $|t| > T$ for some $T < \infty$
  - lots of examples!
- $g(\cdot)$ band-limited to $[-W, W]$ if:
  $G(f) = 0$ for all $|f| > W$ for some $W < \infty$
  - male speech limited to 8000 Hz (= cycles/second)
  - orchestra limited to 20,000 Hz
  - has representation

$$g(t) \overset{\text{ms}}{=} \int_{-W}^{W} G(f) e^{i2\pi f t}\, df$$

  - can be differentiated arbitrary number of times
  - very "smooth"
- can $g(\cdot)$ be both time- and band-limited?
- of considerable interest (Chapters 6 and 7):
  time-limited sequences that are close to band-limited

Similarity Theorem

- first of three reciprocity relationships
  (“ground level/underworld”, p. 66, from Bracewell)
- $g(\cdot) \longleftrightarrow G(\cdot)$ implies
  $|a|^{1/2} g(at) \longleftrightarrow \frac{1}{|a|^{1/2}} G(f/a)$
- for $a > 1$, $|a|^{1/2} g(at)$ formed by
  - contracting $g(\cdot)$ horizontally
  - expanding $g(\cdot)$ vertically
- example: Figure 71 ($a = 1, 2, 4$)

$$g(t) = \frac{1}{(2\pi)^{1/2}} e^{-t^2/2}, \qquad G(f) = e^{-2\pi^2 f^2}$$

Equivalent Width

- measures concentration of signal in time
- best if $g(\cdot)$ real, nonnegative, even, cont. at 0
- definition:

$$\text{width}_{\text{e}}\,\{g(\cdot)\} \equiv \int_{-\infty}^{\infty} g(t)\,dt/g(0)$$

- width of rectangular signal whose
  - height is $g(0)$
  - area is area under curve of $g(\cdot)$
- see Figure 72
- note: area $= \int_{-\infty}^{\infty} g(t)\,dt = G(0)$ & $g(0) = \int_{-\infty}^{\infty} G(f)\,df$
- implies

$$\text{width}_{\text{e}}\,\{g(\cdot)\} = G(0)/\int_{-\infty}^{\infty} G(f)\,df = \frac{1}{\text{width}_{\text{e}}\,\{G(\cdot)\}}$$

- product of widths of signal & transform = unity

Fundamental Uncertainty Relationship

- if $g(\cdot)$ real & nonnegative with unit area, then $g(\cdot)$ is probability density function (pdf)
- consider uniform pdf $r(\cdot; \mu_r, W_r)$: centered at μ_r, width $2W_r$, height $1/2W_r$
- variance measures spread of pdf:

$$\sigma_r^2 \equiv \int_{-\infty}^{\infty} (t - \mu_r)^2 \, r(t; \mu_r, W_r) \, dt = \frac{W_r^2}{3}$$

- relate "natural width" $2W_r$ & σ_r^2: $2W_r = 2\sigma_r\sqrt{3}$
- if $g(\cdot)$ has nonunit area, form $\tilde{g}(t)$ (same width!):

$$\tilde{g}(t) \equiv g(t) / \int_{-\infty}^{\infty} g(t) \, dt$$

- define $\mathrm{width_v}\,\{g(\cdot)\} \equiv 2\sigma_{\tilde{g}}\sqrt{3}$
- for general $g(\cdot)$, use $\mathrm{width_v}\,\{|g(\cdot)|^2\}$
- suppose $|g(\cdot)|^2$ integrates to unity (i.e., is a pdf):

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt = 1 = \int_{-\infty}^{\infty} |G(f)|^2 \, df$$

- let σ_g^2, σ_G^2 be variances of $|g(\cdot)|^2$, $|G(\cdot)|^2$
- can show (pp. 73–4): $\sigma_g^2 \times \sigma_G^2 \geq 1/16\pi^2$ note: equality holds only in Gaussian case!

Convolution Theorem

- briefly: convolution in time domain
  same as multiplication in frequency domain
- convolution of $g(\cdot)$ & $h(\cdot)$ is this function of t:

$$\int_{-\infty}^{\infty} g(u)h(t-u)\,du \equiv g*h(t)$$

  - assumes integral exists
  - "reflect and translate" second function ($h(\cdot)$)
  - $g*h(\cdot)$ notation for function defined above
  - change of variable shows $h*g(\cdot)$ same as $g*h(\cdot)$
- Fourier transform of $g*h(\cdot)$ (see p. 82; Exercise [3.8]):

$$\int_{-\infty}^{\infty} g*h(t)e^{-i2\pi ft}\,dt = G(f)H(f)$$

- $g*h(\cdot) \longleftrightarrow G(\cdot)H(\cdot)$
- variety of convolution theorems in literature (stipulate conditions for above to hold)

Convolution as Smoothing Operation

- regard $g(\cdot)$ as a signal; $h(\cdot)$ as a smoother (filter)
- can regard $g * h(\cdot)$ as smoothed version of $g(\cdot)$
- example
  - a signal: $g(t) = \sum_{l=1}^{L} A_l \cos(2\pi f_l t + \phi_l)$,
  - a smoother: $h(t) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-t^2/(2\sigma^2)}$
    (σ is adjustable smoothing parameter)
  - smoothed version of $g(\cdot)$ (pp. 83–4):

$$g * h(t) = \sum_{l=1}^{L} e^{-(\sigma 2\pi f_l)^2/2} A_l \cos(2\pi f_l t + \phi_l)$$

    * frequencies, phases unchanged
    * $0 < e^{-(\sigma 2\pi f_l)^2/2} < 1$ is attenuation factor
    * smoother shrinks amplitudes toward 0
    * as $f_l \to 0$, attenuation factor increases to 1
    * as $f_l \to \infty$, attenuation factor decreases to 0
    * reduces amplitudes of high frequency terms

Specific Examples of Smoothing

- example with $f_1 = 1/6$ and $f_2 = 3$ (Figure 83):

$$g(t) = 5\cos(2\pi \frac{1}{6} t + 0.5) + \cos(2\pi 3t + 1.1)$$

- attenuation factors:

	$f_1 = 1/6$	$f_2 = 3$
$\sigma = 0.1$	0.99	0.17
$\sigma = 0.25$	0.97	0.0
$\sigma = 0.625$	0.81	0.0

- another smoother: $r(t) = \begin{cases} 1/2\delta, & -\delta \le t \le \delta; \\ 0, & \text{otherwise.} \end{cases}$

- smoothed version of $g(\cdot)$ (note: $\text{sinc}(u) \equiv \sin(\pi u)/(\pi u)$):

$$g*r(t) = \frac{1}{2\delta}\int_{t-\delta}^{t+\delta} g(u)\,du = \sum_{l=1}^{L} \text{sinc}\,(2f_l\delta) A_l \cos(2\pi f_l t + \phi_l)$$

- $\text{sinc}\,(2f_l\delta)$ varies about 0 (not monotonic in f_l)

- example (Figure 86):
  - $\delta = 1/6$ eliminates f_2 term completely
  - $\delta = 1/4$ causes ripples!

- prefer smoothers with monotonic attenuation

Cross and Autocorrelations

- variations on convolution idea
- cross-correlation of $g(\cdot)$ and $h(\cdot)$:

$$g^* \star h(t) \equiv \int_{-\infty}^{\infty} g^*(u)h(u+t)\,du$$

- can show (Exercise [3.7a]): $g^* \star h(\cdot) \longleftrightarrow G^*(\cdot)H(\cdot)$
- letting $h(\cdot) = g(\cdot)$ yields autocorrelation:

$$g^* \star g(t) \equiv \int_{-\infty}^{\infty} g^*(u)g(u+t)\,du$$

- have $g^* \star g(\cdot) \longleftrightarrow G^*(\cdot)G(\cdot) = |G(\cdot)|^2$
- leads to another measure of width:

$$\begin{aligned} \text{width}_{\text{a}}\{g(\cdot)\} &\equiv \text{width}_{\text{e}}\{g^* \star g(\cdot)\} \\ &= \frac{\int_{-\infty}^{\infty} g^* \star g(t)\,dt}{g^* \star g(0)} \\ &\overset{\text{why?}}{=} \frac{\left|\int_{-\infty}^{\infty} g(t)\,dt\right|^2}{\int_{-\infty}^{\infty} |g(t)|^2\,dt} \end{aligned}$$

(will prove useful in Chapter 6)

Disc. Time/Cont. Frequency: I

- assumptions
  - $g(\cdot)$: finite energy; cont. at $t\,\Delta t,\ t = 0, \pm 1, \ldots$
  - samples of $g(\cdot)$: $g_t \equiv g(t\,\Delta t)$
    note: $\Delta t > 0$ is time interval between samples
  - sequence $\{g_t\}$ square summable: $\sum_{t=-\infty}^{\infty} |g_t|^2 < \infty$
- definition: discrete Fourier transform of $\{g_t\}$ is

$$G_p(f) \equiv \Delta t \sum_{t=-\infty}^{\infty} g_t e^{-i2\pi f t\,\Delta t}$$

  - first motivation: if $g(\cdot) \longleftrightarrow G(\cdot)$, then

$$\begin{aligned} G(f) &\equiv \int_{-\infty}^{\infty} g(t) e^{-i2\pi f t}\,dt \\ &\approx \Delta t \sum_{t=-\infty}^{\infty} g(t\,\Delta t) e^{-i2\pi f t\,\Delta t} = G_p(f) \end{aligned}$$

  - second motivation: use Dirac delta functions (p. 88)
- reason for p subscript (note $e^{-i2\pi t} = 1$ for integer t):

$$\begin{aligned} G_p(f + \tfrac{1}{\Delta t}) &= \Delta t \sum_{t=-\infty}^{\infty} g_t e^{-i2\pi (f + \frac{1}{\Delta t}) t\,\Delta t} \\ &= \Delta t \sum_{t=-\infty}^{\infty} g_t e^{-i2\pi f t\,\Delta t} e^{-i2\pi t} = G_p(f) \end{aligned}$$

  $G_p(\cdot)$ is periodic with period $T = 1/\Delta t$ (deja vu!)

Disc. Time/Cont. Frequency: II

- apply cont. time/disc. freq. theory to $G_p(\cdot)$
- Fourier coefficients for $G_p(\cdot)$ are, say,

$$\begin{aligned}\tilde{g}_n &\equiv \frac{1}{T}\int_{-T/2}^{T/2} G_p(t)e^{-i2\pi f_n t}dt \text{ with } f_n = \frac{n}{T} = n\,\Delta t \\ &= \Delta t \int_{-1/2\Delta t}^{1/2\Delta t} G_p(t)e^{-i2\pi tn\,\Delta t}\,dt\end{aligned}$$

- Fourier synthesis of $G_p(\cdot)$ is thus

$$G_p(t) = \sum_{n=-\infty}^{\infty} \tilde{g}_n e^{i2\pi f_n t} = \sum_{n=-\infty}^{\infty} \tilde{g}_n e^{i2\pi tn\,\Delta t}$$

- changing (i) n to t and (ii) t to f yields

$$\begin{aligned}\tilde{g}_t &= \Delta t \int_{-1/2\Delta t}^{1/2\Delta t} G_p(f)e^{-i2\pi ft\,\Delta t}\,df \\ G_p(f) &= \sum_{t=-\infty}^{\infty} \tilde{g}_t e^{i2\pi ft\,\Delta t}\end{aligned}$$

- letting $\tilde{g}_t = g_{-t}\Delta t$ yields

$$\begin{aligned}g_t &= \int_{-1/2\Delta t}^{1/2\Delta t} G_p(f)e^{i2\pi ft\,\Delta t}\,df \\ G_p(f) &= \Delta t \sum_{t=-\infty}^{\infty} g_t e^{-i2\pi ft\,\Delta t}\end{aligned}$$

  2nd equation is definition; 1st gives inverse DFT

- notation: $\{g_t\} \longleftrightarrow G_p(\cdot)$
- Parseval etc. falls out readily

Two Questions of Interest

- given just $g_{-m}, \ldots, g_m$ (a finite sample), how well can $G_p(\cdot)$ be approximated?
- how are $G(\cdot)$ and $G_p(\cdot)$ related?
- answers to questions involve discussion of:
  - leakage, convergence factors (windows)
  - aliasing

Finite Sample Approximation of $G_p(\cdot)$

- assuming $\Delta t = 1$, can approximate $G_p(\cdot)$ using

$$\begin{aligned} G_{p,m}(f) &= \sum_{t=-m}^{m} g_t e^{-i2\pi ft} \\ &= \sum_{t=-m}^{m} \left(\int_{-1/2}^{1/2} G_p(f') e^{i2\pi f' t}\, df' \right) e^{-i2\pi ft} \\ &= \int_{-1/2}^{1/2} G_p(f') \left(\sum_{t=-m}^{m} e^{i2\pi (f'-f)t} \right) df' \\ &\overset{[1.3]}{=} (2m+1) \int_{-1/2}^{1/2} G_p(f') \mathcal{D}_{2m+1}(f - f')\, df', \end{aligned}$$

  - $\mathcal{D}_{2m+1}(\cdot)$ is Dirichlet's kernel; i.e., $\propto$ FT of "rectangular" sequence $\{r_t\}$
  - example of "inverse" convolution theorem: $\{g_t \times r_t\} \longleftrightarrow (2m+1) G_p * \mathcal{D}_{2m+1}(\cdot)$
  - approximation best in least squares sense

- Figure 91 shows $\mathcal{D}_{2m+1}(\cdot)$ for $m = 4, 16, 64$
  - ideally would like to have Dirac δ function (why?)
  - central lobe smears out features (Figure 92) (loss of resolution due to finite sample of data)
  - sidelobes cause leakage and "Gibbs" (Figure 93); (note: some sidelobes are negative!)

- can reduce leakage & Gibbs using Cesàro sums

Cesàro Sums – I

- let $\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots$ be an infinite sequence
- form mth partial sum: $s_m \equiv \sum_{t=-m}^{m} u_t$
- form average of partial sums of orders $0, \ldots, m-1$:

$$a_m \equiv \frac{1}{m} \sum_{j=0}^{m-1} s_j = \sum_{t=-m}^{m} \left(1 - \frac{|t|}{m}\right) u_t$$

  (to see this, work out what a_m is for, e.g., $m = 3$)
- above called (two-sided) Cesàro sum
- theorem: if $s_m \to s$, then $a_m \to s$ also
- application: let

$$s_m = \sum_{t=-m}^{m} g_t e^{-i2\pi ft} = G_{p,m}(f)$$

- since $G_{p,m}(f) \to G_p(f)$, must also have

$$G_{p,m}^{(C)}(f) \equiv \sum_{t=-m}^{m} \left(1 - \frac{|t|}{m}\right) g_t e^{-i2\pi ft} \to G_p(f)$$

  i.e., $G_{p,m}^{(C)}(\cdot)$ is another approximation for $G_p(\cdot)$

Cesàro Sums – II

- claim:

$$\begin{aligned} G^{(C)}_{p,m}(f) &\equiv \sum_{t=-m}^{m} \left(1 - \frac{|t|}{m}\right) g_t e^{-i2\pi f t} \\ &= m^2 \int_{-1/2}^{1/2} G_p(f') \mathcal{D}^2_m(f' - f)\, df' \end{aligned}$$

- sketch of proof:
  - $(1 - |t|/m) \propto$ convolution of $\{r_t\}$ with itself
  - FT of $\{r_t\} \propto \mathcal{D}_m(\cdot)$
  - thus FT of $(1 - |t|/m) \propto \mathcal{D}^2_m(\cdot)$
  - FT of $(1 - |t|/m) \times g_t =$ convolution of FTs
- $\mathcal{D}^2_m(\cdot)$ related to Fejér's kernel (Chapter 6)
- comparing $\mathcal{D}_{2m+1}(\cdot)$ (Fig. 91) to $\mathcal{D}^2_m(\cdot)$ (Fig. 95)
  - $\mathcal{D}^2_m(\cdot)$ has smaller sidelobes (hurray!)
  - $\mathcal{D}^2_m(\cdot)$ has nonnegative sidelobes (hurray!)
  - $\mathcal{D}^2_m(\cdot)$ has wider central lobe (boo!)
- compare Figures 96 and 93 to see tradeoffs
- more generally, approximate $G_p(\cdot)$ using $\sum_{t=-m}^{m} c_t g_t e^{-i2\pi f t}$ ($\{c_t\}$'s are convergence factors or windows)

Relating $G_p(\cdot)$ to $G(\cdot)$: Aliasing

- assume general Δt (i.e., $\Delta t \neq 1$ necessarily)
- $\{g_t\} \longleftrightarrow G_p(\cdot)$ implies $g_t = \int_{-1/(2\,\Delta t)}^{1/(2\,\Delta t)} G_p(f) e^{i2\pi f t\,\Delta t}\,df$
- $g(\cdot) \longleftrightarrow G(\cdot)$ and $g_t = g(t\,\Delta t)$ imply

$$\begin{aligned} g_t &= \int_{-\infty}^{\infty} G(f') e^{i2\pi f' t\,\Delta t}\,df' \\ &= \sum_{k=-\infty}^{\infty} \int_{(2k-1)/(2\,\Delta t)}^{(2k+1)/(2\,\Delta t)} G(f') e^{i2\pi f' t\,\Delta t}\,df' \\ &= \sum_{k=-\infty}^{\infty} \int_{-1/(2\,\Delta t)}^{1/(2\,\Delta t)} G(f + k/\Delta t) e^{i2\pi (f+k/\Delta t) t\,\Delta t}\,df, \end{aligned}$$

  where $f \equiv f' - k/\Delta t$
- note: $e^{i2\pi (f+k/\Delta t) t\,\Delta t} = e^{i2\pi f t\,\Delta t} e^{i2\pi k t} = e^{i2\pi f t\,\Delta t}$, so

$$g_t = \int_{-1/(2\,\Delta t)}^{1/(2\,\Delta t)} \left(\sum_{k=-\infty}^{\infty} G(f + k/\Delta t) \right) e^{i2\pi f t\,\Delta t}\,df$$

- equating integrands: $G_p(f) = \sum_{k=-\infty}^{\infty} G(f + k/\Delta t)$
- holds for $|f| \leq 1/2\Delta t \equiv f_{(N)} \equiv$ Nyquist frequency
- $f \pm k/\Delta t,\ \ k \neq 0$, are *aliases* of f (see Figures 98–9)
- highest f that is not an alias of a lower freq. is $f_{(N)}$
- when can we recover $G(\cdot)$ perfectly from $G_p(\cdot)$?

Disc./Cont. Concentration Problem

- assumptions ($\Delta t = 1$ for convenience)
  - g_t real-valued & time-limited to $t = 0, \ldots, N-1$
  - $\{g_t\} \longleftrightarrow G_p(\cdot)$
- energy $= \sum_{t=0}^{N-1} g_t^2 = \int_{-1/2}^{1/2} |G_p(f)|^2 \, df$
- how close can $G_p(\cdot)$ be to bandlimited?
- for $0 < W < 1/2$, consider concentration measure:

$$\beta^2(W) \equiv \int_{-W}^{W} |G_p(f)|^2 \, df \Big/ \int_{-1/2}^{1/2} |G_p(f)|^2 \, df$$

- reduction of numerator (using $|z|^2 = zz^*$):

$$\begin{aligned}
\int_{-W}^{W} |G_p(f)|^2 \, df &= \int_{-W}^{W} \left(\sum_{t=0}^{N-1} g_t e^{-i2\pi f t} \right) \left(\sum_{t'=0}^{N-1} g_{t'} e^{i2\pi f t'} \right) df \\
&= \sum_{t=0}^{N-1} \sum_{t'=0}^{N-1} g_t g_{t'} \int_{-W}^{W} e^{i2\pi f (t'-t)} \, df \\
&= \sum_{t=0}^{N-1} \sum_{t'=0}^{N-1} g_t g_{t'} \frac{\sin [2\pi W (t'-t)]}{\pi (t'-t)}
\end{aligned}$$

- $\mathbf{g} \equiv [g_0, \ldots, g_{N-1}]^T$
- $A =$ matrix, (t', t)th element $= \sin [2\pi W (t'-t)] / \pi (t'-t)$
- $\beta^2(W) = \mathbf{g}^T A \mathbf{g} / \mathbf{g}^T \mathbf{g}$

Solution to Concentration Problem

- to maximize $\beta^2(W)$, differentiate wrt $\mathbf{g}$: $\frac{d\beta^2(W)}{d\mathbf{g}} = \mathbf{0}$
- solution $\mathbf{g}$ satisfies $A\mathbf{g} = \beta^2(W)\mathbf{g}$
- eigenvalue/eigenvector problem: $A\mathbf{g} = \lambda\mathbf{g}$
- solution is eigenvector, say $\mathbf{v}_0(N, W)$, associated with largest eigenvalue, say, $\lambda_0(N, W) = \beta^2(W)$
- $\mathbf{v}_0(N, W)$ is (subsequence) of 0th order dpss
- A is positive definite, so all N eigenvalues are positive
- can show that eigenvalues are distinct, so order as:

  $0 < \lambda_{N-1}(N, W) < \cdots < \lambda_1(N, W) < \lambda_0(N, W) < 1$

  why must $\lambda_0(N, W)$ be less than unity?
- first $2NW$ (Shannon number) eigenvalues close to 1, after which $\lambda_k(N, W)$'s fall off rapidly to 0
- can use eigenvectors to form orthonormal basis:

$$\mathbf{v}_j(N, W)^T \mathbf{v}_k(N, W) = \begin{cases} 1, & j = k; \\ 0, & \text{otherwise.} \end{cases}$$

- see Figures 106–10

Disc. Time/Disc. Frequency: I

- $\{g_t : t = 0, \ldots, N-1\}$, sampled Δt units apart
- two possible definitions for Fourier transform:
  - form infinite sequence: set $g_t \equiv 0$ for other t's:

    $$G_p(f) = \Delta t \sum_{t=-\infty}^{\infty} g_t e^{-i2\pi f t\,\Delta t} = \Delta t \sum_{t=0}^{N-1} g_t e^{-i2\pi f t\,\Delta t}$$

    useful (e.g., periodogram), but infinite # of f's
  - define discrete Fourier transform (DFT) of $\{g_t\}$:

    $$G_n \equiv G_p(f_n) = \Delta t \sum_{t=0}^{N-1} g_t e^{-i2\pi f_n t\,\Delta t} = \Delta t \sum_{t=0}^{N-1} g_t e^{-i2\pi nt/N},$$

    where f_n are N Fourier (standard) frequencies:

    $$f_n \equiv n/N\,\Delta t, \qquad n = 0, 1, \ldots, N-1$$
- inverse DFT derived as follows:

$$\begin{aligned}\sum_{n=0}^{N-1} G_n e^{i2\pi nt'/N} &= \Delta t \sum_{n=0}^{N-1} \sum_{t=0}^{N-1} g_t e^{i2\pi n(t'-t)/N} \\ &= \Delta t \sum_{t=0}^{N-1} g_t \sum_{n=0}^{N-1} e^{i2\pi n(t'-t)/N} \overset{[1.3]}{=} \Delta t\, g_{t'} N\end{aligned}$$

- inverse DFT: $g_t = \dfrac{1}{N\,\Delta t} \sum_{n=0}^{N-1} G_n e^{i2\pi nt/N}$
- fast Fourier transform (FFT) algorithm vs. DFT
- two standard forms: $\Delta t = 1$ or $\Delta t = 1/N$

Disc. Time/Disc. Frequency: II

- DFT definition implies $g_t = 0$ beyond $t = 0$ to $N-1$
- notion of *zero padding*
  - add $N' - N > 0$ zeros to $g_0, \ldots, g_{N-1}$ to form $g_0, \ldots, g_{N'-1}$ with $g_t = 0$, $N \leq t \leq N' - 1$
  - DFT of padded sequence ($f'_n \equiv n/N'\,\Delta t$):
    $$G'_n \equiv \Delta t \sum_{t=0}^{N'-1} g_t e^{-i2\pi nt/N'} = \Delta t \sum_{t=-\infty}^{\infty} g_t e^{-i2\pi nt/N'} = G_p(f'_n)$$
    evaluates $G_p(\cdot)$ over finer grid than f_n's
  - useful to compute convolutions or DFT via chirp transform algorithm (p. 114)
- can also claim DFT implies periodic extension:
  $$g_t \equiv \frac{1}{N\,\Delta t} \sum_{n=0}^{N-1} G_n e^{i2\pi nt/N}, \quad t < 0 \text{ or } t \geq N,$$
  so $\{g_t : t = 0, \pm 1, \pm 2, \ldots\}$ has period N
- can use RHS of DFT to define G_n for all n, so
  $$g_t = \frac{1}{N\,\Delta t} \sum_{n=k}^{N+k-1} G_n e^{i2\pi nt/N}$$
  for any integer k
- summary of Fourier theory, pp. 116–9