Spectral Representation Theorem: I

• $|\text{Fourier transform}|^2$ of $\{g_t\}$ decomposes energy across $f$’s (Parseval)

• goal: decompose variance of $\{X_t\}$ across frequencies (assume $E\{X_t\} = 0$, $\Delta t = 1$)

• $\text{var} \{X_t\} = E\left\{\frac{1}{N} \sum_{i=1}^{N} X_i^2\right\} = \text{expected power}$

• can we use “$\{g_t\}$” theory to represent $\{X_t\}$?

• Cramér (1942) used infinite-order harmonic processes

• start with finite-order harmonic process:

$$X_t = \sum_{l=1}^{L} D_l \cos (2\pi f_l t + \phi_l), \quad t = 0, \pm 1, \pm 2, \ldots$$

– $D_l$’s and $f_l$’s real-valued constants

– $\phi_l$ terms independent rv’s, uniformly distributed over $[-\pi, \pi]$

– $f_l$ ordered such that $f_l < f_{l+1}$

– $0 < f_l < 1/2$ for all $l$ (simplifies treatment)
Spectral Representation Theorem: II

- rewrite cosine terms as complex exponentials:
  \[ D_l \cos (2\pi f_l t + \phi_l) = \frac{D_l}{2} \left( e^{i\phi_l} e^{i2\pi f_l t} + e^{-i\phi_l} e^{-i2\pi f_l t} \right) \]

- can reexpress model using complex exponentials:
  \[ X_t = \sum_{l=-L}^{L} C_l e^{i2\pi f_l t}, \quad C_l = \begin{cases} D_l e^{i\phi_l}/2, & l = 1, \ldots, L; \\ D_0 \equiv 0, & l = 0; \\ C_{-l}^*, & l = -L, \ldots, -1, \end{cases} \]
  where \( f_0 \equiv 0 \) and \( f_{-l} \equiv -f_l \)

- \( C_1, \ldots, C_L \) pairwise uncorrelated (why?)

- Exer. [4.1]: \( \text{cov}\{C_{-l}, C_l\} = 0 \Rightarrow C_l \)'s uncorrelated!

- yields \( \text{var}\{X_t\} = \sum_{l=-L}^{L} \text{var}\{C_l e^{i2\pi f_l t}\} \)
  
  - \( \text{var}\{C_l e^{i2\pi f_l t}\} = |e^{i2\pi f_l t}|^2 \text{var}\{C_l\} = \text{var}\{C_l\} \)
  
  - \( \text{var}\{C_l\} = E\{|C_l|^2\} - |E\{C_l\}|^2 \)
  
  - \( E\{|C_l|^2\} = E\{|D_l e^{i\phi_l}/2|^2\} = D_l^2/4 \)
  
  - \( E\{C_l\} = E\{D_l e^{i\phi_l}/2\} = D_l E\{e^{i\phi_l}\} = 0 \)

- yields \( \text{var}\{X_t\} = \sum_{l=-L}^{L} D_l^2/4 \)
Spectral Representation Theorem: III

- can define variance spectrum:

\[ S^{(V)}(f) \equiv \begin{cases} \frac{D^2_l}{4}, & \text{if } f = f_l \text{ for } l = -L, \ldots, L; \\ 0, & \text{otherwise}. \end{cases} \]

- define complex-valued “jump” process on \([0, 1/2]\):

\[ Z(f) \equiv \begin{cases} 0, & \text{if } f = 0; \\ \sum_{j=0}^l C_j, & \text{for } f_l < f \leq f_{l+1}, \; l = 0, \ldots, L. \end{cases} \]

Note: \( f_{L+1} \equiv 1/2 \)

- implies, for example:

\[ Z(f) = \begin{cases} 0, & \text{if } 0 \leq f \leq f_1; \\ C_1, & \text{if } f_1 < f \leq f_2; \\ C_1 + C_2, & \text{if } f_2 < f \leq f_3; \\ C_1 + C_2 + C_3, & \text{if } f_3 < f \leq f_4; \text{ etc.} \end{cases} \]

- for small \( df > 0 \), define increment process:

\[ dZ(f) \equiv \begin{cases} Z(f + df) - Z(f), & \text{if } 0 \leq f \& f + df < 1/2; \\ 0, & \text{if } f = 1/2; \\ dZ^*(-f), & \text{if } -1/2 \leq f < 0. \end{cases} \]

- note: \( dZ(f_l) = Z(f_l + df) - Z(f_l) = C_l \)

while \( dZ(f) = 0 \) if \( f \neq f_l \) for some \( l \)

- \( E\{dZ(f)\} = 0 \& \text{ var } \{dZ(f)\} = S^{(V)}(f) \) for all \( f \)
Spectral Representation Theorem: IV

• \{Z(f)\} called process with orthogonal increments:
  \[
  \text{cov}\{dZ(f'), dZ(f)\} = \begin{cases} 
  E\{|C_l|^2\} = S^{(V)}(f), & f' = f = f_l; \\
  0, & \text{otherwise}. 
\end{cases}
  \]

• let \(g(\cdot)\) be a continuous function defined on \([-1/2, 1/2]\), & let \(H(\cdot)\) be piecewise constant with jumps \(b_1, b_2, \ldots, b_N\) at locations \(-1/2 < a_1 < a_2 < \cdots < a_N < 1/2\)

• by definition \(\int_{-1/2}^{1/2} g(f) \, dH(f) \equiv \sum_{k=1}^{N} g(a_k)b_k\)

• thus: \(X_t = \sum_{l=-L}^{L} C_le^{i2\pi ft} = \int_{-1/2}^{1/2} e^{i2\pi ft} \, dZ(f)\)

• RHS is spectral representation for harmonic process

• by letting \(L \to \infty\), holds for all real-valued discrete parameter stationary processes \(\{X_t\}\)

• properties of \(\{Z(f)\}\) in general case:
  - \(E\{dZ(f)\} = 0\) for all \(|f| \leq 1/2\)
  - var \(\{dZ(f)\} = E\{|dZ(f)|^2\} = dS^{(I)}(f)\), where \(S^{(I)}(\cdot)\) is integrated spectrum
  - \(\text{cov}\{dZ(f'), dZ(f)\} = 0\) for \(f' \neq f\)
Basic Consequences of Theorem

• integrated spectrum determines acvs:

\[ s_\tau = E\{X_t X_{t+\tau}\} = E\{X_t^* X_{t+\tau}\} \]

\[ = E\left\{ \int_{-1/2}^{1/2} e^{-i2\pi f' t} dZ^* (f') \int_{-1/2}^{1/2} e^{i2\pi f(t+\tau)} dZ (f) \right\} \]

\[ = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} e^{i2\pi (f-f') t} e^{i2\pi f \tau} E\{dZ^* (f') dZ (f)\} \]

\[ = \int_{-1/2}^{1/2} e^{i2\pi f \tau} E\{|dZ (f)|^2\} = \int_{-1/2}^{1/2} e^{i2\pi f \tau} dS^{(I)} (f) \]

• if \( S^{(I)}(\cdot) \) differentiable, then \( dS^{(I)}(f) = S(f) \, df \)
  (note: need not be differentiable)

• \( S(\cdot) \) called spectral density function (sdf):

\[ s_\tau = \int_{-1/2}^{1/2} e^{i2\pi f \tau} S(f) \, df \]

• if \( S(\cdot) \) square integrable, can appeal to FT theory:

\[ S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f \tau} \]

• sdf sometimes \emph{defined} as above (not true in general)

• when true, have \( \{ s_\tau \} \longleftrightarrow S(\cdot) \)
Extension to Other Stationary Processes

- real-valued cont. parameter stat. process \{X(t)\}:
  \[ X(t) = \int_{-\infty}^{\infty} e^{i2\pi ft} \, dZ(f) \]
  - only limits of integration change
  - \( s(\tau) = \int_{-\infty}^{\infty} e^{i2\pi f\tau} \, dS^{(I)}(f) = \int_{-\infty}^{\infty} S(f)e^{i2\pi f\tau} \, df \)
    if \( S^{(I)}(\cdot) \) is differentiable
  - \( S(f) = \int_{-\infty}^{\infty} s(\tau)e^{-i2\pi f\tau} \, d\tau \)
    if \( S(\cdot) \) square integrable

- complex-valued stationary processes \{Z_t\} & \{Z(t)\}:
  \[ dZ(f) = dZ^*(-f) \text{ need not hold (only change!)} \]
Basic Properties of Spectrum

- \( E\{|dZ(f)|^2\} = dS^{(I)}(f) = S(f) \, df \Rightarrow S(f) \geq 0 \)
- \( dZ(f) = dZ^*(-f) \Rightarrow S(f) = S(-f) \); i.e., “2 sided”
- \( S(f) = \frac{dS^{(I)}(f)}{df} \Rightarrow S^{(I)}(f) = \int_{-1/2}^{f} S(f') \, df' \)
- implies \( S^{(I)}(-1/2) = 0 \)
- if \( f_1 < f_2 \), then \( S^{(I)}(f_1) \leq S^{(I)}(f_2) \) (nondecreasing)
- \( s_0 = \int_{-1/2}^{1/2} dS^{(I)}(f) = S^{(I)}(f) \bigg|_{-1/2}^{1/2} = S^{(I)}(1/2); \)
- \( S^{(I)}(1/2) = \text{var} \{X_t\} \)
- \( S^{(I)}(\cdot) \) bounded because \( \text{var} \{X_t\} < \infty \) always
- \( 0 \leq S^{(I)}(f) \leq s_0 = \text{var} \{X_t\} \)
- \( \text{var} \{X_t\} = s_0 = \int_{-1/2}^{1/2} S(f) \, df \)
- Wold’s theorem: necessary and sufficient condition for \( \{s_\tau\} \) to be acvs for some stat. process \( \{X_t\} \) is existence of nondecreasing function \( S^{(I)}(\cdot) \) such that \( S^{(I)}(-1/2) = 0, \ S^{(I)}(1/2) = \text{var} \{X_t\} \) and
  \[ s_\tau = \int_{-1/2}^{1/2} e^{i2\pi f \tau} \, dS^{(I)}(f) \]
Example: Spectrum for White Noise

- let \( \{\epsilon_t\} \) be white noise with variance \( \sigma^2 \)
- acvs given by \( s_0 = \sigma^2 \) & \( s_\tau = 0 \) for \( \tau \neq 0 \)
- consider \( S^{(I)}(f) = \sigma^2(f + \frac{1}{2}) \)
  - nondecreasing function of \( f \), as required
  - \( S^{(I)}(-1/2) = 0 \), as required
  - \( S^{(I)}(1/2) = \sigma^2 \), as required
- differentiable with derivative \( S(f) = \sigma^2 \)
- note that
  \[
  \int_{-1/2}^{1/2} e^{i2\pi f\tau} dS^{(I)}(f) = \int_{-1/2}^{1/2} e^{i2\pi f\tau} S(f) df = \sigma^2 \int_{-1/2}^{1/2} e^{i2\pi f\tau} df = \begin{cases} \sigma^2, & \tau = 0; \\ 0, & \tau \neq 0, \end{cases}
  \]
  \( = s_\tau \)
- Wold’s theorem \( \Rightarrow S^{(I)}(\cdot) \) integrated spectrum
- \( S(\cdot) \) is sdf for \( \{\epsilon_t\} \)
- sdf is constant (analogous to white light)
  (\( \{X_t\} \) with nonconstant sdf = “colored” noise)
Example: Harmonic Process

- acvs given by $s_\tau = \sum_{l=1}^{L} \frac{D_l^2}{2} \cos(2\pi f_l \tau) = \sum_{l=-L}^{L} \frac{D_l^2}{4} e^{i2\pi f_l \tau}$
  (here $D_0 = f_0 = 0$; $D_{-l} = D_l$; $f_{-l} = -f_l$)

- let $S^{(I)}(\cdot)$ be step function, jumps at $\pm f_l$ of size $D_l^2/4$
  - nondecreasing function of $f$, as required
  - $S^{(I)}(-1/2) = 0$, as required
  - $S^{(I)}(1/2) = \sum_{l=-L}^{L} \frac{D_l^2}{4} = \sum_{l=1}^{L} \frac{D_l^2}{2} = s_0$, as required

- note that
  $$\int_{-1/2}^{1/2} e^{i2\pi f \tau} dS^{(I)}(f) = \sum_{l=-L}^{L} \frac{D_l^2}{4} e^{i2\pi f_l \tau} = s_\tau$$

- Wold’s theorem $\Rightarrow$ $S^{(I)}(\cdot)$ integrated spectrum

- $S^{(I)}(\cdot)$ not differentiable $\Rightarrow$ no true sdf, but:
  - can define “sdf” using Dirac $\delta$ functions:
    $$S(f) = \sum_{l=-L}^{L} \frac{D_l^2}{4} \delta(f - f_l)$$
  - integration yields step function:
    $$\int_{-1/2}^{f} S(f') df' = \sum_{l: f_l \leq f} \frac{D_l^2}{4} = S^{(I)}(f)$$
Classification of Spectra

• sdf $S(\cdot)$ resembles pdf \textit{except}: sdf integrates to process variance ($\neq 1$ in general)
• $S^{(I)}(\cdot)$ resembles probability distribution function
• Lebesgue decomposition theorem (adapted to $S^{(I)}(\cdot)$): can always write $S^{(I)}(\cdot)$ as a sum of up to three canonical integrated spectra $S_1^{(I)}(\cdot), S_2^{(I)}(\cdot)$ & $S_3^{(I)}(\cdot)$, each of which is nondecreasing and satisfies $S_i^{(I)}(-1/2) = 0$

1. $S_1^{(I)}(\cdot)$ is absolutely continuous, so
$$\frac{dS_1^{(I)}(f)}{df} = S(f), \text{ an sdf}$$

2. $S_2^{(I)}(\cdot)$ is a step function with jump of size $p_l > 0$
   at $f_l, l = 1, 2, \ldots$

3. $S_3^{(I)}(\cdot)$ is a continuous singular function:
   – derivative is zero almost everywhere
   – continuous
   – strictly increasing
   – bizarre!: can ignore in practical applications
Four Classes of Spectra

1. purely continuous: $S^{(I)}(f) = S_1^{(I)}(f)$
   - $\{X_t\}$ has sdf $S(\cdot)$ equivalent to $S^{(I)}(\cdot)$
   - Riemann–Lebesgue: $s_\tau \to 0$ as $|\tau| \to \infty$
   - examples: white noise, AR($p$) & MA($q$) processes

2. purely discrete: $S^{(I)}(f) = S_2^{(I)}(f)$
   - $\{X_t\}$ has line spectrum equivalent to $S^{(I)}(\cdot)$
   - acvs doesn’t converge to zero as $|\tau| \to \infty$
   - $\{X_t\}$ is a harmonic process

3. discrete: $S^{(I)}(f) = \sigma^2 (f + \frac{1}{2}) + S_2^{(I)}(f)$ with $\sigma^2 > 0$
   - $S_1^{(I)}(\cdot) \Leftrightarrow$ sdf for white noise
   - acvs doesn’t converge to zero as $|\tau| \to \infty$
   - $\{X_t\}$ is a harmonic process + white noise

4. mixed: $S^{(I)}(f) = S_1^{(I)}(f) + S_2^{(I)}(f)$
   - $S_1^{(I)}(\cdot) \Leftrightarrow$ sdf for colored noise
   - acvs doesn’t converge to zero as $|\tau| \to \infty$
   - $\{X_t\}$ is a harmonic process + colored noise
   - examples: Figures 142–3
Sampling and Aliasing

• suppose \( \{X(t)\} \) has sdf \( s(\cdot) \iff S_{X(t)}(\cdot) \)

• define \( X_t = X(t_0 + t \Delta t) \)

• acvs for \( \{X_t\} \) given by

\[
s_\tau = \text{cov} \{X_t, X_{t+\tau}\}
= \text{cov} \{X(t_0 + t \Delta t), X(t_0 + [t + \tau] \Delta t)\} = s(\tau \Delta t)
\]

• sdf for \( \{X_t\} \) given by p. 98:

\[
S_{X_t}(f) = \sum_{k=-\infty}^{\infty} S_{X(t)}(f+k/\Delta t), \quad |f| \leq \frac{1}{2 \Delta t} \equiv f(N)
\]

\( S_{X_t}(\cdot) \) is aliased version of \( S_{X(t)}(\cdot) \)

• example: Figure 145

\(- \Delta t = 1/2 \) preferable to \( \Delta t = 1/4 \)?