Spectral Representation Theorem: I

- |Fourier transform|^2 of $\{g_t\}$ decomposes energy across f's (Parseval)
- goal: decompose variance of $\{X_t\}$ across frequencies (assume $E\{X_t\} = 0,\ \ \Delta t = 1$)
- $\text{var}\,\{X_t\} = E\{\frac{1}{N}\sum_{t=1}^{N} X_t^2\} =$ expected power
- can we use "$\{g_t\}$" theory to represent $\{X_t\}$?
- Cramér (1942) used infinite-order harmonic processes
- start with finite-order harmonic process:

$$X_t = \sum_{l=1}^{L} D_l \cos\left(2\pi f_l t + \phi_l\right), \qquad t = 0, \pm 1, \pm 2, \ldots$$

  - D_l's and f_l's real-valued constants
  - ϕ_l terms independent rv's, uniformly distributed over $[-\pi, \pi]$
  - f_l ordered such that $f_l < f_{l+1}$
  - $0 < f_l < 1/2$ for all l (simplifies treatment)

Spectral Representation Theorem: II

- rewrite cosine terms as complex exponentials:

$$D_l \cos(2\pi f_l t + \phi_l) = \frac{D_l}{2}\left(e^{i\phi_l} e^{i2\pi f_l t} + e^{-i\phi_l} e^{-i2\pi f_l t}\right)$$

- can reexpress model using complex exponentials:

$$X_t = \sum_{l=-L}^{L} C_l e^{i2\pi f_l t}, \qquad C_l = \begin{cases} D_l e^{i\phi_l}/2, & l = 1, \ldots, L; \\ D_0 \equiv 0, & l = 0; \\ C_{-l}^{*}, & l = -L, \ldots, -1, \end{cases}$$

  where $f_0 \equiv 0$ and $f_{-l} \equiv -f_l$

- $C_1, \ldots, C_L$ pairwise uncorrelated (why?)
- Exer. [4.1]: $\mathrm{cov}\{C_{-l}, C_l\} = 0 \Rightarrow C_l$'s uncorrelated!
- yields $\mathrm{var}\{X_t\} = \sum_{l=-L}^{L} \mathrm{var}\{C_l e^{i2\pi f_l t}\}$
  - $\mathrm{var}\{C_l e^{i2\pi f_l t}\} = |e^{i2\pi f_l t}|^2 \,\mathrm{var}\{C_l\} = \mathrm{var}\{C_l\}$
  - $\mathrm{var}\{C_l\} = E\{|C_l|^2\} - |E\{C_l\}|^2$
  - $E\{|C_l|^2\} = E\{|D_l e^{i\phi_l}/2|^2\} = D_l^2/4$
  - $E\{C_l\} = E\{\frac{D_l}{2} e^{i\phi_l}\} = \frac{D_l}{2} E\{e^{i\phi_l}\} = 0$
- yields $\mathrm{var}\{X_t\} = \sum_{l=-L}^{L} D_l^2/4$

Spectral Representation Theorem: III

- can define variance spectrum:

$$S^{(V)}(f) \equiv \begin{cases} D_l^2/4, & \text{if } f = f_l \text{ for } l = -L, \ldots, L; \\ 0, & \text{otherwise.} \end{cases}$$

- define complex-valued "jump" process on $[0, 1/2]$:

$$Z(f) \equiv \begin{cases} 0, & \text{if } f = 0; \\ \Sigma_{j=0}^{l} C_j, & \text{for } f_l < f \leq f_{l+1},\ l = 0, \ldots, L. \end{cases}$$

note: $f_{L+1} \equiv 1/2$

- implies, for example:

$$Z(f) = \begin{cases} 0, & \text{if } 0 \leq f \leq f_1; \\ C_1, & \text{if } f_1 < f \leq f_2; \\ C_1 + C_2, & \text{if } f_2 < f \leq f_3; \\ C_1 + C_2 + C_3, & \text{if } f_3 < f \leq f_4; \text{ etc.} \end{cases}$$

- for small $df > 0$, define increment process:

$$dZ(f) \equiv \begin{cases} Z(f + df) - Z(f), & \text{if } 0 \leq f \ \&\ f + df < 1/2; \\ 0, & \text{if } f = 1/2; \\ dZ^*(-f), & \text{if } -1/2 \leq f < 0. \end{cases}$$

- note: $dZ(f_l) = Z(f_l + df) - Z(f_l) = C_l$
  while $dZ(f) = 0$ if $f \neq f_l$ for some l

- $E\{dZ(f)\} = 0$ & $\text{var}\,\{dZ(f)\} = S^{(V)}(f)$ for all f

Spectral Representation Theorem: IV

- $\{Z(f)\}$ called process with orthogonal increments:

  $$\mathrm{cov}\{dZ(f'), dZ(f)\} = \begin{cases} E\{|C_l|^2\} = S^{(V)}(f), & f' = f = f_l; \\ 0, & \text{otherwise.} \end{cases}$$

- let $g(\cdot)$ be a continuous function defined on $[-1/2, 1/2]$, & let $H(\cdot)$ be piecewise constant with jumps $b_1, b_2, \ldots, b_N$ at locations $-1/2 < a_1 < a_2 < \cdots < a_N < 1/2$

- by definition $\int_{-1/2}^{1/2} g(f)\, dH(f) \equiv \sum_{k=1}^{N} g(a_k) b_k$

- thus: $X_t = \sum_{l=-L}^{L} C_l e^{i2\pi f_l t} = \int_{-1/2}^{1/2} e^{i2\pi f t}\, dZ(f)$

- RHS is spectral representation for harmonic process

- by letting $L \to \infty$, holds for *all* real-valued discrete parameter stationary processes $\{X_t\}$

- properties of $\{Z(f)\}$ in general case:
  - $E\{dZ(f)\} = 0$ for all $|f| \leq 1/2$
  - $\mathrm{var}\,\{dZ(f)\} = E\{|dZ(f)|^2\} = dS^{(I)}(f)$, where $S^{(I)}(\cdot)$ is *integrated spectrum*
  - $\mathrm{cov}\,\{dZ(f'), dZ(f)\} = 0$ for $f' \neq f$

Basic Consequences of Theorem

- integrated spectrum determines acvs:

$$\begin{aligned} s_\tau &= E\{X_t X_{t+\tau}\} = E\{X_t^* X_{t+\tau}\} \\ &= E\left\{\int_{-1/2}^{1/2} e^{-i2\pi f' t}\, dZ^*(f') \int_{-1/2}^{1/2} e^{i2\pi f(t+\tau)}\, dZ(f)\right\} \\ &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{i2\pi(f-f')t} e^{i2\pi f\tau} E\{dZ^*(f')\, dZ(f)\} \\ &= \int_{-1/2}^{1/2} e^{i2\pi f\tau} E\{|dZ(f)|^2\} = \int_{-1/2}^{1/2} e^{i2\pi f\tau}\, dS^{(I)}(f) \end{aligned}$$

- if $S^{(I)}(\cdot)$ differentiable, then $dS^{(I)}(f) = S(f)\, df$ (note: need not be differentiable)

- $S(\cdot)$ called spectral density function (sdf):

$$s_\tau = \int_{-1/2}^{1/2} e^{i2\pi f\tau}\, S(f)\, df$$

- if $S(\cdot)$ square integrable, can appeal to FT theory:

$$S(f) = \sum_{\tau=-\infty}^{\infty} s_\tau e^{-i2\pi f\tau}$$

- sdf sometimes *defined* as above (not true in general)

- when true, have $\{s_\tau\} \longleftrightarrow S(\cdot)$

Extension to Other Stationary Processes

- real-valued cont. parameter stat. process $\{X(t)\}$:

$$X(t) = \int_{-\infty}^{\infty} e^{i2\pi ft}\, dZ(f)$$

  - only limits of integration change
  - $s(\tau) = \int_{-\infty}^{\infty} e^{i2\pi f\tau}\, dS^{(I)}(f) = \int_{-\infty}^{\infty} S(f)e^{i2\pi f\tau}\, df$
    if $S^{(I)}(\cdot)$ is differentiable
  - $S(f) = \int_{-\infty}^{\infty} s(\tau)e^{-i2\pi f\tau}\, d\tau$
    if $S(\cdot)$ square integrable

- complex-valued stationary processes $\{Z_t\}$ & $\{Z(t)\}$:
  $dZ(f) = dZ^*(-f)$ need not hold (only change!)

Basic Properties of Spectrum

- $E\{|dZ(f)|^2\} = dS^{(I)}(f) = S(f)\,df \Rightarrow S(f) \geq 0$
- $dZ(f) = dZ^*(-f) \Rightarrow S(f) = S(-f)$; i.e., "2 sided"
- $S(f) = \dfrac{dS^{(I)}(f)}{df} \Rightarrow S^{(I)}(f) = \int_{-1/2}^{f} S(f')\,df'$
- implies $S^{(I)}(-1/2) = 0$
- if $f_1 < f_2$, then $S^{(I)}(f_1) \leq S^{(I)}(f_2)$ (nondecreasing)
- $s_0 = \int_{-1/2}^{1/2} dS^{(I)}(f) = S^{(I)}(f)\Big|_{-1/2}^{1/2} = S^{(I)}(1/2)$;
- $S^{(I)}(1/2) = \mathrm{var}\,\{X_t\}$
- $S^{(I)}(\cdot)$ bounded because $\mathrm{var}\,\{X_t\} < \infty$ always
- $0 \leq S^{(I)}(f) \leq s_0 = \mathrm{var}\,\{X_t\}$
- $\mathrm{var}\,\{X_t\} = s_0 = \int_{-1/2}^{1/2} S(f)\,df$
- Wold's theorem: necessary and sufficient condition for $\{s_\tau\}$ to be acvs for some stat. process $\{X_t\}$ is existence of nondecreasing function $S^{(I)}(\cdot)$ such that $S^{(I)}(-1/2) = 0$, $S^{(I)}(1/2) = \mathrm{var}\,\{X_t\}$ and

$$s_\tau = \int_{-1/2}^{1/2} e^{i2\pi f\tau}\,dS^{(I)}(f)$$

Example: Spectrum for White Noise

- let $\{\epsilon_t\}$ be white noise with variance σ^2
- acvs given by $s_0 = \sigma^2$ & $s_\tau = 0$ for $\tau \neq 0$
- consider $S^{(I)}(f) = \sigma^2(f + \frac{1}{2})$
  - nondecreasing function of f, as required
  - $S^{(I)}(-1/2) = 0$, as required
  - $S^{(I)}(1/2) = \sigma^2$, as required
- differentiable with derivative $S(f) = \sigma^2$
- note that

$$\begin{aligned}\int_{-1/2}^{1/2} e^{i2\pi f\tau}\, dS^{(I)}(f) &= \int_{-1/2}^{1/2} e^{i2\pi f\tau} S(f)\, df \\ &= \sigma^2 \int_{-1/2}^{1/2} e^{i2\pi f\tau}\, df = \begin{cases} \sigma^2, & \tau = 0; \\ 0, & \tau \neq 0, \end{cases} \\ &= s_\tau\end{aligned}$$

- Wold's theorem $\Rightarrow S^{(I)}(\cdot)$ integrated spectrum
- $S(\cdot)$ is sdf for $\{\epsilon_t\}$
- sdf is constant (analogous to white light)
  ($\{X_t\}$ with nonconstant sdf = "colored" noise)

Example: Harmonic Process

- acvs given by $s_\tau = \sum_{l=1}^{L} \frac{D_l^2}{2} \cos(2\pi f_l \tau) = \sum_{l=-L}^{L} \frac{D_l^2}{4} e^{i2\pi f_l \tau}$ (here $D_0 = f_0 = 0$; $D_{-l} = D_l$; $f_{-l} = -f_l$)
- let $S^{(I)}(\cdot)$ be step function, jumps at $\pm f_l$ of size $D_l^2/4$
  - nondecreasing function of f, as required
  - $S^{(I)}(-1/2) = 0$, as required
  - $S^{(I)}(1/2) = \sum_{l=-L}^{L} \frac{D_l^2}{4} = \sum_{l=1}^{L} \frac{D_l^2}{2} = s_0$, as required
- note that
$$\int_{-1/2}^{1/2} e^{i2\pi f\tau}\, dS^{(I)}(f) = \sum_{l=-L}^{L} \frac{D_l^2}{4} e^{i2\pi f_l \tau} = s_\tau$$
- Wold's theorem $\Rightarrow$ $S^{(I)}(\cdot)$ integrated spectrum
- $S^{(I)}(\cdot)$ not differentiable $\Rightarrow$ no true sdf, **but**:
  - can define "sdf" using Dirac δ functions:
$$S(f) = \sum_{l=-L}^{L} \frac{D_l^2}{4} \delta(f - f_l)$$
  - integration yields step function:
$$\int_{-1/2}^{f} S(f')\, df' = \sum_{l:f_l \le f} \frac{D_l^2}{4} = S^{(I)}(f)$$

Classification of Spectra

- sdf $S(\cdot)$ resembles pdf *except*:
  sdf integrates to process variance ($\neq 1$ in general)
- $S^{(I)}(\cdot)$ resembles probability distribution function
- Lebesgue decomposition theorem (adapted to $S^{(I)}(\cdot)$): can always write $S^{(I)}(\cdot)$ as a sum of up to three canonical integrated spectra $S_1^{(I)}(\cdot)$, $S_2^{(I)}(\cdot)$ & $S_3^{(I)}(\cdot)$, each of which is nondecreasing and satisfies
  $S_i^{(I)}(-1/2) = 0$

  1. $S_1^{(I)}(\cdot)$ is absolutely continuous, so
     $$\frac{dS_1^{(I)}(f)}{df} = S(f), \text{ an sdf}$$
  2. $S_2^{(I)}(\cdot)$ is a step function with jump of size $p_l > 0$ at f_l, $l = 1, 2, \ldots$
  3. $S_3^{(I)}(\cdot)$ is a continuous singular function:
     - derivative is zero almost everywhere
     - continuous
     - strictly increasing
     - bizarre!: can ignore in practical applications

Four Classes of Spectra

1. purely continuous: $S^{(I)}(f) = S_1^{(I)}(f)$
   - $\{X_t\}$ has sdf $S(\cdot)$ equivalent to $S^{(I)}(\cdot)$
   - Riemann–Lebesgue: $s_\tau \to 0$ as $|\tau| \to \infty$
   - examples: white noise, AR(p) & MA(q) processes
2. purely discrete: $S^{(I)}(f) = S_2^{(I)}(f)$
   - $\{X_t\}$ has *line spectrum* equivalent to $S^{(I)}(\cdot)$
   - acvs doesn't converge to zero as $|\tau| \to \infty$
   - $\{X_t\}$ is a harmonic process
3. discrete: $S^{(I)}(f) = \sigma^2(f + \frac{1}{2}) + S_2^{(I)}(f)$ with $\sigma^2 > 0$
   - $S_1^{(I)}(\cdot) \Leftrightarrow$ sdf for white noise
   - acvs doesn't converge to zero as $|\tau| \to \infty$
   - $\{X_t\}$ is a harmonic process + white noise
4. mixed: $S^{(I)}(f) = S_1^{(I)}(f) + S_2^{(I)}(f)$
   - $S_1^{(I)}(\cdot) \Leftrightarrow$ sdf for colored noise
   - acvs doesn't converge to zero as $|\tau| \to \infty$
   - $\{X_t\}$ is a harmonic process + colored noise

- examples: Figures 142–3

Sampling and Aliasing

- suppose $\{X(t)\}$ has sdf $s(\cdot) \longleftrightarrow S_{X(t)}(\cdot)$
- define $X_t = X(t_0 + t\,\Delta t)$
- acvs for $\{X_t\}$ given by

$$\begin{aligned} s_\tau &= \operatorname{cov}\{X_t, X_{t+\tau}\} \\ &= \operatorname{cov}\{X(t_0 + t\,\Delta t), X(t_0 + [t+\tau]\,\Delta t)\} = s(\tau\,\Delta t) \end{aligned}$$

- sdf for $\{X_t\}$ given by p. 98:

$$S_{X_t}(f) = \sum_{k=-\infty}^{\infty} S_{X(t)}(f + k/\Delta t), \qquad |f| \le \frac{1}{2\,\Delta t} \equiv f_{(N)}$$

$S_{X_t}(\cdot)$ is aliased version of $S_{X(t)}(\cdot)$

- example: Figure 145
  - $\Delta t = 1/2$ preferable to $\Delta t = 1/4$?