Exercise 4 of Assignment 1 (due 1/14/08)

This exercise illustrates the point that the concept of stationarity is properly defined only for models and not for data. We do so by claiming that, given any observed time series $\{x_t : t = 0, 1, \ldots, N-1\}$, we can always construct a stochastic process $\{X_t : t = 0, 1, \ldots, N-1\}$ that is stationary and has $\{x_t\}$ as one of its realizations, with the observed series having a nonzero probability of being selected. To establish this claim, define the ensemble for $\{X_t\}$ to consist of N realizations given by $\{x_t\}$ and all possible circular shifts. For example, if $N = 5$, the ensemble consists of

$$\begin{array}{l} \{x_0, x_1, x_2, x_3, x_4\}, \\ \{x_1, x_2, x_3, x_4, x_0\}, \\ \{x_2, x_3, x_4, x_0, x_1\}, \\ \{x_3, x_4, x_0, x_1, x_2\} \text{ and} \\ \{x_4, x_0, x_1, x_2, x_3\}. \end{array}$$

If we index the realizations in the ensemble by $k = 0, 1, \ldots, N-1$, we can mathematically describe the kth realization as $\{x_{t+k \bmod N} : t = 0, 1, \ldots, N-1\}$, where, for any integer l, we define '$l \bmod N$' as follows. If $0 \le l \le N-1$, then $l \bmod N = l$; otherwise, $l \bmod N = l + mN$, where m is the unique integer such that $0 \le l + mN \le N-1$. For example, when $N = 5$ and $k = 2$,

$$\begin{aligned} \{x_{t+2 \bmod 5} : t = 0, 1, \ldots, 4\} &= \{x_{2 \bmod 5}, x_{3 \bmod 5}, x_{4 \bmod 5}, x_{5 \bmod 5}, x_{6 \bmod 5}\} \\ &= \{x_2, x_3, x_4, x_0, x_1\}. \end{aligned}$$

To complete our definition of $\{X_t\}$, we stipulate that the probability of picking any given realization in the ensemble is $1/N$. Thus, if κ is a random variable that takes on the values $0, 1, \ldots, N-1$ with equal probability, we can express the stochastic process $\{X_t\}$ as $\{x_{t+\kappa \bmod N}\}$. Show that, for all $0 \le s, t \le N-1$,

(a) $E\{X_t\}$ is a constant that does not depend on t and

(b) $\text{cov}\{X_s, X_t\}$ is a constant that depends upon just the lag $s - t$.

Thus $\{X_t\}$ is a stationary process with mean $\mu = E\{X_t\}$ and an ACVS given by $s_\tau = \text{cov}\{X_\tau, X_0\}$ for $-(N-1) \le \tau \le N-1$.