Exercise 4 of Assignment 4 (due 2/4/08)

Suppose that $\{X_t\}$ is a real-valued stationary process whose sampling interval is Δt. Assume that its acvs $\{s_{X,\tau}\}$ and sdf $S_X(\cdot)$ are related as dictated by Equations (134a) and (134b):

$$\begin{aligned} s_{X,\tau} &= \int_{-f_{(N)}}^{f_{(N)}} S_X(f) e^{i2\pi f\tau\,\Delta t}\,df, \quad \tau = \ldots, -1, 0, 1, \ldots, \\ S_X(f) &= \Delta t \sum_{\tau=-\infty}^{\infty} s_{X,\tau} e^{-i2\pi f\tau\,\Delta t}, \quad |f| \le f_{(N)} \equiv \frac{1}{2\,\Delta t}, \end{aligned}$$

where $f_{(N)}$ is the Nyquist frequency. Note that we can use the last equation above to define $S_X(\cdot)$ for $|f| > f_{(N)}$, so that the sdf can be regarded as a periodic function with period $2f_{(N)}$. Let $Y_t = X_{2t}$, $t = \ldots, -1, 0, 1, \ldots$; i.e., the process $\{Y_t\}$ is formed by subsampling every other random variable from the process $\{X_t\}$, and hence the sampling interval for $\{Y_t\}$ is $\Delta t' \equiv 2\,\Delta t$.

a) Show that $\{Y_t\}$ is a stationary process, and determine its acvs $\{s_{Y,\tau}\}$ in terms of $\{s_{X,\tau}\}$.

b) Show that $\{Y_t\}$ has an sdf $S_Y(\cdot)$ that is an aliased version of $S_X(\cdot)$, with the two sdfs being related by

$$S_Y(f) = S_X(f) + S_X(f + f_{(N)}).$$

Argue that $S_Y(\cdot)$ is a periodic function with a period of $2f'_{(N)}$, where $f'_{(N)} \equiv \frac{1}{2\,\Delta t'}$ is the Nyquist frequency for $\{Y_t\}$.

c) Suppose now that $\{X_t\}$ is a first order moving average process with mean zero and coefficient θ; i.e., we can write $X_t = \epsilon_t - \theta\epsilon_{t-1}$, where $\{\epsilon_t\}$ is a white noise process with mean zero and variance σ_ϵ^2. The acvs and sdf for this process are given by

$$s_{X,\tau} = \begin{cases} \sigma_\epsilon^2(1+\theta^2), & \tau = 0; \\ -\sigma_\epsilon^2\theta, & \tau = \pm 1; \\ 0, & |\tau| \ge 2; \end{cases} \quad \text{and} \quad S_X(f) = \sigma_\epsilon^2\,\Delta t \left|1 - \theta e^{-i2\pi f\,\Delta t}\right|^2$$

(see pages 43 and 167). Use part a of this exercise to show that $\{Y_t\}$ has an acvs of a white noise process. Use part b to show that $\{Y_t\}$ has an sdf of a white noise process.