Appendix A: The Capital Asset Pricing Model (CAPM) and Beta

(REFERENCES: Any of the Introductory Investment textbooks)

- Investors Use Mean Variance Optimal Portfolio’s
  - Rules for how to invest (normative economics)

- Capital Asset Pricing Model (CAPM)
  - An equilibrium description of how assets are priced (positive economics)

- Main Implications of CAPM
  - Expected return of an asset is related to a measure of risk called “Beta”
A.1 The CAPM Result

Single period return for asset $i$: $r_i$

Single period return for the market: $r_M$

Single period risk-free rate of return: $r_f$

\[ E(r_i) - r_f = \beta_i \cdot \left( E(r_M) - r_f \right) \]

where

\[ \beta_i = \frac{\text{cov}(r_i, r_M)}{\text{var}(r_M)} \]
Two Ways to View CAPM

View 1:

\[ E(r_i) - r_f \] as a (straight line) function of \( E(r_M) - r_f \) with slope \( \beta_i \) fixed (sometimes called the characteristic line)

This is a simple linear model with predictor \( E(r_M) - r_f \)

View 2:

\[ E(r_i) - r_f \] as a (straight line) function of \( \beta_i \) with slope \( E(r_M) - r_f \) fixed.

This is a simple linear model with predictor \( \beta_i \). The resulting straight-line plot is called the security market line.
In terms of **excess** expected returns \( \mu_i = E(r_i) \) and 
\( \mu_M = E(r_M) \):

\[
\mu_i = \beta \cdot \mu_M
\]

This is linear prediction/regression **through the origin**

CAPM implies that there are no additional expected returns returns for an asset other than that explained by expected market returns and beta, i.e., \( \alpha = 0 \) in the linear model with intercept:

\[
\mu_i = \alpha_i + \beta_i \cdot \mu_M
\]

**Active Portfolio Management:** The attempt to take advantage of failure of CAPM because \( \alpha \neq 0 \)
A.2 Portfolio Returns and Beta

Portfolio expected returns:

\[ E(r_p) = \sum_{i=1}^{n} w_i \cdot E(r_i) \]

Use the CAPM for each asset to get

\[ E(r_i) - r_f = \beta_i \cdot (E(r_M) - r_f) \]

to get

\[ E(r_p) - r_f = \beta_p (E(r_M) - r_f) \quad \text{where} \quad \beta_p = \sum_{i=1}^{n} w_i \cdot \beta_i \]

Note also that:

\[ \beta_p = \sum_{i=1}^{n} w_i \cdot \frac{\text{cov}(r_i, r_M)}{\sigma_M^2} = \frac{\text{cov} \left( \sum_{i=1}^{n} w_i \cdot r_i, r_M \right)}{\sigma_M^2} = \frac{\text{cov}(r_p, r_M)}{\sigma_M^2} \]
A.3 Portfolio Risk and CAPM Beta’s

Use the single-factor market model for each stock $i$:

$$ r_{i,t} = \alpha + \beta_i \cdot r_{M,t} + \varepsilon_{i,t}, \quad t = 1, 2, \ldots, T, \quad i = 1, 2, \ldots, n $$

Assume that errors and market returns are uncorrelated, errors for each stock have zero mean with constant variance $\sigma_{i,\varepsilon}^2$, and the market returns have constant variance $\sigma_M^2$. No assumptions about temporal or cross-sectional correlations among the errors for now. Then:

$$ \sigma_i^2 = \beta_i^2 \cdot \sigma_M^2 + \sigma_{i,\varepsilon}^2 \quad i = 1, 2, \ldots, n $$

**WARNING:** This model is generally quite inadequate, and we will need additional predictor variables (factors) to obtain a good model.
\[ r_P = \sum_{i=1}^{n} w_i \cdot r_i \]

\[ = \sum_{i=1}^{n} w_i \cdot \alpha + \sum_{i=1}^{n} w_i \cdot \beta_i \cdot r_M + \sum_{i=1}^{n} w_i \cdot \varepsilon_i \]

\[ = \alpha + \beta_P \cdot r_M + \sum_{i=1}^{n} w_i \cdot \varepsilon_i \]

where \[ \beta_P = \sum_{i=1}^{n} w_i \cdot \beta_i \]

So:

\[
\sigma_P^2 = \beta_P^2 \cdot \sigma_M^2 + \text{var}\left(\sum_{i=1}^{n} w_i \cdot \varepsilon_i\right)
\]

Copyright R. Douglas Martin
A.4 Estimating CAPM Beta’s

- Uniform Industry Practice with Least Squares
  - Compute LS fit of the market model (see Appendix 3A.2)
  - Beta = $\hat{\beta}_{LS}$ (raw slope coefficient)
  - Adjusted Beta: $\hat{\beta}_{ADJ} = .33 + .67 \cdot \hat{\beta}_{LS}$ (Bayesian and frequentist rationale)

- Modern Robust Alternative to Least Squares
  - Returns contain outliers that distort LS Beta
  - Use robust regression (not much influenced by outliers)
  - Will show how to do this in S-PLUS later
Least Squares (LS) Fit of Market Model

You learned in a statistics course that the least squares fitting problem

$$\arg \min_{\alpha, \beta} \sum_{t=1}^{T} (r_t - \alpha - \beta \cdot r_{M,t})^2$$

has as a solution the least squares (LS) coefficient estimates

$$\hat{\alpha} = \bar{r} - \hat{\beta} \cdot \bar{r}_M$$

$$\hat{\beta} = \frac{\sum_{t=1}^{T} (r_t - \bar{r})(r_{M,t} - \bar{r}_M)}{\sum_{t=1}^{T} (r_{M,t} - \bar{r}_M)^2}$$

**A W COMPUTER SYSTEMS INC.**

<table>
<thead>
<tr>
<th>Method</th>
<th>Beta</th>
<th>T-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>ROBUST</td>
<td>1.1</td>
<td>(0.33)</td>
</tr>
<tr>
<td>OLS</td>
<td>2.33</td>
<td>(1.13)</td>
</tr>
</tbody>
</table>

**OIL CITY PETROLEUM INC.**

<table>
<thead>
<tr>
<th>Method</th>
<th>Beta</th>
<th>T-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>ROBUST</td>
<td>0.86</td>
<td>(0.47)</td>
</tr>
<tr>
<td>OLS</td>
<td>3.27</td>
<td>(0.9)</td>
</tr>
</tbody>
</table>

**CHIEF CONSOLIDATED MNG CO**

<table>
<thead>
<tr>
<th>Method</th>
<th>Beta</th>
<th>T-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>ROBUST</td>
<td>0.5</td>
<td>(0.26)</td>
</tr>
<tr>
<td>OLS</td>
<td>1.12</td>
<td>(0.8)</td>
</tr>
</tbody>
</table>

**METALLURGICAL INDUSTRIES INC.**

<table>
<thead>
<tr>
<th>Method</th>
<th>Beta</th>
<th>T-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>ROBUST</td>
<td>1.14</td>
<td>(0.22)</td>
</tr>
<tr>
<td>OLS</td>
<td>2.05</td>
<td>(1.62)</td>
</tr>
</tbody>
</table>