1. (30 points). Define five of the following eight terms:
   (a) Absolute continuity of a signed measure \( \phi \) with respect to a measure \( \mu \), and singularity of \( \phi \) with respect to \( \mu \).
   (b) The product \( \sigma \)-field \( A \times A' \) for two measurable spaces \((\Omega, A)\) and \((\Omega', A')\).
   (c) Almost sure convergence of a sequence of random variables \( \{X_n\} \).
   (d) Independent random variables \( X_1, \ldots, X_n \) and independent events \( A_1, \ldots, A_n \).
   (e) The tail \( \sigma \)-field of a sequence of random variables \( X_1, X_2, \ldots \).
   (f) A \( \pi \)-system \( C \).
   (g) A \( \lambda \)-system \( D \).
   (h) Khintchine - equivalent sequences of random variables.

2. (30 points). Give careful statements of three of the following six theorems or results:
   (a) The first Borel-Cantelli lemma.
   (b) The Kolmogorov zero-one law.
   (c) Feller’s weak law of large numbers.
   (d) The strong law of large numbers.
   (e) The \( \pi - \lambda \) theorem.
   (f) Fatou’s lemma.
3. (30 points). Let $X : \Omega \to \mathbb{R}$ be a random variable defined on the probability space $(\Omega, \mathcal{A}, P)$, let $P_X$ denote the induced distribution of $X$ on $(\mathbb{R}, \mathcal{B})$, and let $g$ be a measurable function from $\mathbb{R}$ to $\mathbb{R}$.

(a) State the theorem of the unconscious statistician in this context.

(b) Sketch a proof of the theorem you stated in (a).

4. (30 points) Suppose that $X$ and $Y$ are independent random variables and that $f$ and $g$ are real-valued measurable functions from $(\mathbb{R}, \mathcal{B})$ to $(\mathbb{R}, \mathcal{B})$ such that $f(X)$ and $g(Y)$ are measurable. Suppose that $E|f(X)| < \infty$ and $E|g(Y)| < \infty$. Show that

$$E[f(X)g(Y)] = E[f(X)]E[g(Y)] \quad (1)$$

Do either 5 or 6:

5. (30 points). Suppose that $X, X_1, X_2, \ldots$ are independent and identically distributed random variables.

(a) Show that the following identities holds: for all $\lambda > 0$

$$P(\max_{1 \leq k \leq n} |X_k| > \lambda) = P(|X| > \lambda) \sum_{k=1}^{n} P(|X| \leq \lambda)^{k-1} = 1 - P(|X| \leq \lambda)^n.$$

[Hint: For the first identity use the same type of decomposition of the event on the left side as we used in the proof of Kolmogorov’s inequality.]

(b) Use the identities in (a) to show that for $\epsilon > 0$

$$P(\max_{1 \leq k \leq n} |X_k| > n\epsilon) \begin{cases} \leq nP(|X| > n\epsilon) \\ \geq 1 - \exp(-nP(|X| > n\epsilon)). \end{cases}$$

(c) Use the results of (b) to show that $M_n \equiv n^{-1}\max_{1 \leq k \leq n} |X_k| \to_p 0$ if and only if $xP(|X| > x) \to 0$ as $x \to \infty$ (i.e. $X$ is weak-$L_1$).

6. (30 points). Give an example of a distribution function $F$ with density function $f$ with respect to Lebesgue measure $\lambda$ such that $E|X| = \infty$ but $\tau(x) \equiv xP(|X| > x) \to 0$ as $x \to \infty$. Thus if $X_1, \ldots, X_n$ are i.i.d. $F$, the WLLN holds: $\overline{X}_n - \mu_n \to_p 0$ for some sequence $\mu_n$ (where $\mu_n = E(X_11_{|X_1| \leq n})$ works), but the strong law of large numbers fails: $

\limsup_n \overline{|X|} = +\infty \text{ a.s.}$
Do either 7 or 8:

7. (30 points). Suppose that $X_1$ and $X_2$ are independent Rademacher random variables, and set $X_3 = X_1 X_2$. (Thus $P(X_j = \pm 1) = 1/2$ for $j = 1, 2$.)

(a) Show that $X_3$ is a Rademacher random variable: $P(X_3 = \pm 1) = 1/2$.
(b) Show that each pair of $X_1, X_2, X_3$ are independent random variables.
(c) Show that $X_1, X_2, X_3$ are not independent random variables.

8. (30 points). Let $(\Omega, \mathcal{A}, P)$ denote the probability space $([0, 1], \mathcal{B} \cap [0, 1], \lambda)$ where $\lambda$ is Lebesgue measure. For $n = 1, 2, \ldots$ define

\[ X_n(\omega) = \begin{cases} 
1, & \text{if } 0 \leq \omega < 1/3, \\
2, & \text{if } 1/3 \leq \omega < 1/3 + 2/3^n, \\
3, & \text{if } 1/3 + 2/3^n \leq \omega < 1.
\]

(a) Are the $X_n$’s independent?
(b) What is the tail $\sigma$–field of the $X_n$’s?

Do either 9 or 10:

9. (30 points). Let $X_1, X_2, \ldots$ be i.i.d. with d.f. $F(x) = 1 - \exp(-x^\alpha)$ for $x \geq 0$ where $\alpha > 0$.

(a) Find a sequence $b_n$ so that $\limsup_{n \to \infty} (X_n/b_n) = 1$ almost surely.
(b) Let $M_n \equiv \max_{1 \leq k \leq n} X_k$. In the case $\alpha = 1$, find a sequence of numbers $c_n$ so that $M_n - c_n \to^d \text{“something”}$ and find the distribution of “something”.

10. (30 points). Suppose that $X_1, X_2, \ldots$ are uncorrelated and $E(X_j^2) \leq M < \infty$ for all $j \geq 1$.

(a) Show that $\overline{X}_n - E(\overline{X}_n) \to_2 0$.
(b) Show that $\underline{X}_n - E(\underline{X}_n) \to_p 0$.
(c) Show that $n^\alpha (\overline{X}_n - E(\overline{X}_n)) \to_p 0$ for $0 < \alpha < \alpha_0$ for some $\alpha_0$ (and determine $\alpha_0$).