Statistics 521, Problem Set 4
Wellner; 10/19/2016

Reading:
Shorack, PfS, Chapter 3, sections 3.1-3.4, pages 37-61;
Durrett, Probability, sections 1.4-1.6, pages 17-36;
sections 3.1-3.2, pages 94-98.

Reminder: Make-up lecture 1: Wednesday 10/26 12:30 - 1:20, Low 106.
Reminder: Make-up lecture 2: Wednesday 11/16 12:30 - 1:20, Low 106.
Reminder: Midterm exam: Monday, November 14.

Due: Wednesday, October 26, 2007.

1. PfS, Exercise 2.3.4, page 32: (a) Suppose that $\mu(\Omega) < \infty$ and $g$ is continuous a.e. $\mu_X$ (that is, $g$ is continuous except perhaps on a set of $\mu_X$ measure 0). Then $X_n \to_{\mu} X$ implies that $g(X_n) \to_{\mu} g(X)$.
(b) Let $g$ be uniformly continuous on the real line. Then $X_n \to_{\mu} X$ implies that $g(X_n) \to_{\mu} g(X)$. (Here $\mu(\Omega) = \infty$ is allowed.)

2. PfS, Exercise 3.2.1, page 42: Show that $X \geq 0$ and $\int Xd\mu = 0$ implies $\mu([X > 0]) = 0$.

3. PfS, Exercise 3.2.2, page 42: Show that

$$\int_{A} Xd\mu = \begin{cases} 
0, & \text{for all } A \in \mathcal{A} \text{ implies } X = \begin{cases} 
0 \text{ a.e.,} \\
\geq 0 \text{ a.e.}
\end{cases} 
\end{cases}$$

4. PfS, Exercise 3.2.4, page 43. Let $Y \equiv g(X)$ in the context of Theorem 3.2.6 (the “Theorem of the unconscious statistician”). Show that the second equality holds in:

$$\int_{X^{-1}(g^{-1}(B))} g(X(\omega))d\mu(\omega) = \int_{g^{-1}(B)} g(x)d\mu_X(x) = \int_{B} yd\mu_Y(y) \text{ for } B \in \mathcal{B}$$

where $\mu_Y$ is the induced measure of $Y$ on $(\overline{R}, \mathcal{B})$. 
(ii) Suppose that \( \mu \) is Lebesgue measure on the unit interval \([0, 1]\) and that \((a, b) = (0, 1)\) in Exercise 3.3. If \( X(t, \omega) = 1_{[\omega \leq t]} \), then for each \( t \), \( \frac{\partial}{\partial t}X(t, \omega) = 0 \) almost everywhere. But \( \int X(t, \omega) d\mu(\omega) \) does not differentiate to 0. Why is this not a contradiction?

6. **Bonus problem:** (See PfS Example 1.1, page 123; Durrett Example 1.2.4.) The Cantor singular distribution function \( F \) is the function \( F : [0, 1] \rightarrow [0, 1] \) defined as follows: 
\[
F(x) = 1/2 \quad \text{for} \quad x \in (1/3, 2/3);
F(x) = 1/4 \quad \text{for} \quad x \in (1/9, 2/9) \quad \text{and} \quad F(x) = 3/4 \quad \text{for} \quad x \in (7/9, 8/9); \ldots;
F(x) = 1/2^n, 3/2^n, 5/2^n, \ldots \quad \text{on} \quad \text{the successive intervals removed from} \quad C_{n-1} \quad \text{in} \quad \text{the construction of} \quad C_n.
\]
Thus \( F \) is defined on the open set \([0, 1] \setminus C\), is nondecreasing, and has values in \([0, 1]\). Extend it to all of \([0, 1]\) by letting \( F(0) = 0 \), and setting
\[
F(x) \equiv \sup\{F(y) : t \in [0, 1] \setminus C \quad \text{and} \quad y < x\}
\]
for \( x \in C \) and \( x \neq 0 \).
(i) Show that \( F \) is non-decreasing and continuous with \( F(0) = 0 \) and \( F(1) = 1 \). Because \( F \) is continuous, its range is all of \([0, 1]\).
(ii) Now the inverse (or quantile) function \( F^{-1} \) of \( F \) defined by
\[
F^{-1}(y) \equiv \inf\{x \in [0, 1] : F(x) \geq y\}
\]
is one-to-one (injective) and \( F^{-1}([0, 1]) \subset C \). Show that \( F^{-1} \) is Borel-measurable.
(iii) Show that the lengths of the “flat spots” in \( F \) sum to 1.