Reading: Shorack, PfS, Chapter 3, section 3.5, pages 52-63;
Shorack, PfS, Chapter 4, sections 4.1 - 4.4, pages 65-85.
Durrett, Probability, pages 412 - 416.

Due: Wednesday, November 2, 2016.

1. PfS, Exercise 3.4.2, page 48: Show that \( \rho = 1 \) if and only if \( X - \mu_X = a(Y - \mu_Y) \) for some \( a > 0 \); and \( \rho = -1 \) if and only if \( X - \mu_X = a(Y - \mu_Y) \) for some \( a < 0 \). Thus \( \rho \) measures linear dependence, not dependence.

2. PfS, Exercise 3.4.3, page 48: (Littlewood’s inequalities) Let \( \mu_r \equiv E|X|^r \). For \( r \geq s \geq t \geq 0 \) we have \( \mu_r^{s-t} \mu_t^{r-s} \geq \mu_s^{r-t} \). In particular, \( \mu_2^3 \leq \mu_2^2 \mu_4 \).

3. Suppose that \( \epsilon_1, \ldots, \epsilon_n \) are i.i.d. random variables with \( P(\epsilon_i = \pm 1) = 1/2 \), and let \( a_i \in \mathbb{R}, i = 1, \ldots, n \). Khintchine’s inequalities say that for each \( p > 0 \)

\[
A_p \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \leq \left( E|\sum_{i=1}^{n} a_i \epsilon_i|^p \right)^{1/p} \leq B_p \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2}.
\]

for some constants \( A_p \) and \( B_p \). Prove the above inequalities when \( p = 1 \).

**Hint:** The inequality on the right side is easy. Use the previous exercise to prove the inequality on the left side by showing that for \( Z = \sum_{i=1}^{n} a_i \epsilon_i \), we have \( E|Z|^4 \leq 3(E(Z^2))^2 \).

4. PfS, Exercise 3.5.3, page 55: Consider a probability measure \( P \). (a) Let \( Y \geq 0 \) have df \( F \). Show that \( EY = \int_0^{\infty} P(Y \geq y)dy = \int_0^{\infty} [1 - F(y)]dy \).

**[Hint: prove the claimed formula for simple functions by summing by parts; and then the full claim follows from the MCT. A different proof to come later will use Fubini’s theorem.]*

(b) use the result of (a) to show that for \( Y \geq 0 \) and \( \lambda \geq 0 \) we have

\[
\int_{[Y \geq \lambda]} YdP = \lambda P(Y \geq \lambda) + \int_{\lambda}^{\infty} P(Y \geq y)dy.
\]
Draw a picture to illustrate this.
(c) Suppose there is a $Y \in L_1$ such that $P(|X_n| \geq y) \leq P(Y \geq y)$ for all $y > 0$ and all $n \geq 1$. Then use (b) to show that $\{X_n : n \geq 1\}$ is uniformly integrable.

5. (a) Show that if $|X_n| \leq Y$ and $Y$ is integrable, then $\{X_n\}$ is uniformly integrable.
(b) Let $U \sim \text{Uniform}(0,1)$, and let $X_n \equiv (n/\log n)1_{[0,1]}(U)$ for $n \geq 3$. Show that $\{X_n\}$ is uniformly integrable and $\int X_n dP \to 0$ even though they are not dominated by any integrable rv $Y$.
(c) Let $Z_n = n1_{[0,1]}(U) - n1_{[1/n,2/n]}(U)$. Show that $\{Z_n\}$ is not uniformly integrable, but that $\int Z_n dP \to 0$.

6. **Optional bonus problem:** PfS, Exercise 3.4.6, page 50 (qualified by “for all $\epsilon \geq 1$”): Let $T \sim \text{Binomial}(n,p)$, so $P(T = k) = \binom{n}{k}p^k(1-p)^{n-k}$ for $0 \leq k \leq n$. The measure associated with $T$ has mean $np$ and variance $np(1-p)$. Then use inequality 4.6 with $g(x) = \exp(rx)$ and $r > 0$ to show that

$$P(T/n \geq p\epsilon) \leq \exp(-np(h(\epsilon)),$$

where $h(y) \equiv \epsilon(\log(y) - 1) + 1$

for each $\epsilon > 1$. [Hint: It helps to use $T \overset{d}{=} \sum_1^n X_i$ where $X_i \sim \text{Bernoulli}(p)$ are independent, and then apply Theorem 7.1.1 (page 124).]