Statistics 521, Problem Set 8
Wellner; 11/23/2016

Reading: Shorack, PfS, Chapter 7, pages 123 - 129.
Shorack, PfS, Chapter 8, pages 147 - 174.

Due: Wednesday, November 30, 2016.

1. PfS Exercise 6.4.3, page 114. Prove just the parts of these formulas involving $F$, not the parts involving $F^{-1}$. You may also use Fubini’s theorem directly. That is, show that:
   (i) If $X \geq 0$ has d.f. $F$, then
       $$\int_0^\infty P(X > x)dx = E(X) = \int_0^\infty (1 - F(x))dx.$$ 
   (ii) If $E|X| < \infty$ then
       $$E(X) = -\int_{-\infty}^0 F(x)dx + \int_0^\infty (1 - F(x))dx.$$ 
   (iii) Let $r > 0$. If $X \geq 0$, then
       $$\int_0^\infty P(X^r > x)dx = E(X^r) = \int_0^\infty r x^{r-1}(1 - F(x))dx.$$

2. Prove the two formulas in (17), PfS page 113: if $X \geq 0$ is integer valued, then $E(X) = \sum_{k=1}^\infty P(X \geq k)$ and $E(X^2) = \sum_{k=1}^\infty (2k - 1)P(X \geq k)$.

3. PfS Exercise 4.9, page 114: For any distribution function $F$ on $\mathbb{R}$ we have $\int \{F(x + \theta) - F(x)\}dx = \theta$ for each $\theta \geq 0$.

4. PfS Exercise 4.11, page 114:
   (a) Show that $\int_0^\infty \{P(|X| > x)\}^{1/2}dx < \infty$ implies $E(X^2) < \infty$.
   (b) Show that $\int_0^\infty \{P(|X| > x)\}^{1/2}dx \leq \frac{r}{r-2}\|X\|_r$ for any $r > 2$ so that the integral on the left is finite whenever $X \in \mathcal{L}_r$ for any $r > 2$. If $\int_0^\infty \{P(|X| > x)\}^{1/2}dx < \infty$ then we say that $X \in \mathcal{L}_{2,1}$; this condition arises in connection with optimal transportation inequalities for empirical processes and in multiplier and bootstrap CLTs.

Let $\mathcal{X} = [0, 1], \mathcal{Y} = (1, \infty)$ both equipped with the Borel sets and Lebesgue measure. Let $f(x, y) = e^{-xy} - 2e^{-2xy}$. Show that
   (i) $\int_0^1 (\int_1^\infty f(x, y)dy)dx = \int_0^1 x^{-1}(e^{-x} - e^{-2x})dx$ exists and is $> 0$.
   (ii) $\int_1^\infty (\int_0^1 f(x, y)dy)dx = \int_1^\infty y^{-1}(e^{-2y} - e^{-y})dy$ exists and is $< 0$. 

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6. **Optional bonus problem:** Show that \( f(x, y) = e^{-xy} \sin(x) \) is integrable with respect to Lebesgue measure on \( \mathbb{R}^2 \) in the strip \( 0 < x < a, 0 < y \). Perform the double integral in the two different orders to find that

\[
\int_0^a \frac{\sin(x)}{x} dx = \frac{\pi}{2} - \cos(a) \int_0^\infty \frac{e^{-ay}}{1 + y^2} dy - \sin(a) \int_0^\infty \frac{ye^{-ay}}{1 + y^2} dy.
\]

Use the inequality \( 1 + y^2 \geq 1 \) to obtain the bound

\[
\left| \int_0^a \frac{\sin(x)}{x} dx - \frac{\pi}{2} \right| \leq \frac{2}{a}.
\]

Letting \( a \to \infty \) yields \( \int_0^\infty x^{-1} \sin(x) dx = \pi/2 \).

7. **Optional bonus problem:** For \( x \in \mathbb{R} \) and \( t > 0 \) let

\[
f(x, t) \equiv (2\pi t)^{-1/2} \exp\left( -\frac{x^2}{2t} \right).
\]

Let \( g(x, t) \equiv \partial f/\partial t \). Fix \( s > 0 \). Show that

\[
\int_{-\infty}^\infty \int_{-\infty}^\infty g(x, t) dt dx = \int_{-\infty}^\infty -f(x, s) dx = -1,
\]

\[
\int_{-\infty}^\infty \int_{-\infty}^\infty g(x, t) dx dt = \int_{s}^\infty \partial f/\partial x |_{x=\infty} dt = 0.
\]

Hint: Note that \( \partial f/\partial x = -(x/t)f(x, t) \) and \( \partial^2 f/\partial x^2 = (x^2t^{-2} - t^{-1})f = 2g \), and hence that \( f \) satisfies the partial differential equation \( (1/2)\partial^2 f/\partial x^2 = \partial f/\partial t \).

8. **Optional bonus problem:** (A bivariate exponential distribution).

Let \( Y_1 \sim \text{Exponential}(\mu_1), Y_2 \sim \text{Exponential}(\mu_2), \) and \( W \sim \text{Exponential}(\tau) \) be independent.

(a) Let \( X_1 \equiv \min\{Y_1, W\}, X_2 \equiv \min\{Y_2, W\} \). Find the joint survival function \( F(x_1, x_2) \equiv P(X_1 > x_1, X_2 > x_2) \) of \( (X_1, X_2) \).

(b) Find \( P(X_1 = X_2) \).

(c) Use (b) to show that the joint probability distribution \( Q \equiv P_{X_1, X_2} \) of \( (X_1, X_2) \) is not absolutely continuous with respect to Lebesgue measure \( \lambda_2 \) on \( \mathbb{R}^2 \). Identify the singular component of the joint distribution \( Q \) by computing \( P(X_1 = X_2 > z) \) for \( z \geq 0 \).

(d) What is the density of \( Q_{ac} \), the absolutely continuous component of \( Q \) with respect to \( \lambda_2 \)?