1. (28 points). Define four of the following six terms:

(a) The product $\sigma$-field $\mathcal{A} \times \mathcal{A}'$ for two measurable spaces $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$.
(b) Almost sure convergence of a sequence of random variables $\{X_n\}$.
(c) Independent $\sigma$-fields and independent random variables.
(d) The tail $\sigma$-field of a sequence of random variables $X_1, X_2, \ldots$.
(e) Khintchine-equivalent sequences of random variables.
(f) A uniformly integrable sequence of random variables $\{X_n\}$.

Solution: See PfS, Chapters 2-8.

2. (30 points). Give careful statements of three of the following six theorems or results:

(a) The Lebesgue decomposition theorem.
(b) The Kolmogorov zero-one law.
(c) The Fubini-Tonelli theorem.
(d) Kolmogorov’s Strong Law of Large Numbers.
(e) Kolmogorov’s maximal inequality.
(f) Vitali’s theorem (three parts).

Solution: See PfS, Chapters 2-8.

3. (28 points).
(a) State two Borel-Cantelli lemmas.
(b) Prove the first Borel-Cantelli lemma.

Solution: (a) Let $(\Omega, \mathcal{A}, P)$ be a probability space. First Borel-Cantelli Lemma: for any sequence of events or measurable sets $\{A_n\}$, $\sum_{n=1}^{\infty} P(A_n) < \infty$ implies $P(A_n \text{ i.o.}) = 0$. Second Borel-Cantelli Lemma: if the events $\{A_n\}$ are independent, then $\sum_{n=1}^{\infty} P(A_n) = \infty$ implies $P(A_n \text{ i.o.}) = 1$. 


(b) Note that if \( \sum_1^\infty P(A_n) < \infty \), then
\[
P(A_n \text{ i.o.}) = \lim_{n \to \infty} P(A_\infty \cap \bigcup_{m=n}^\infty A_m) = \lim_{n \to \infty} \sum_{m=n}^\infty P(A_m) = 0.
\]

4. (28 points). Suppose that \( X_1, X_2, \ldots \) are independent and identically distributed random variables with \( E|X_1| < \infty \).
Let \( Y_n = X_n1\{|X_n| < n\} \) for \( n = 1, 2, \ldots \).
(a) Show that the sequences \( \{X_n\} \) and \( \{Y_n\} \) are Khintchine - equivalent.
(b) What does the result in (a) imply about \( P(X_n \neq Y_n \text{ i.o.})? \)

**Solution:**
(a) \( \{X_n \neq Y_n\} = \{|X_n| \geq n\} \) so that
\[
\sum_{n=1}^\infty P(X_n \neq Y_n) = \sum_{n=1}^\infty P(|X_n| \geq n) = \sum_{n=1}^\infty P(|X_1| \geq n) \leq E|X_1| < \infty.
\]
Thus the sequences \( \{X_n\} \) and \( \{Y_n\} \) are Khinchine equivalent.
(b) Since the \( \{X_n\}'s \) are independent, the result in (a) together with the Borel-Cantelli lemma imply that \( P(X_n \neq Y_n \text{ i.o.}) = 0 \).

5. (28 points) Suppose that \( X_1, X_2, \ldots \) are i.i.d. with \( E|X_1|^r < \infty \) for some \( r > 0 \). For \( n \geq 1 \) let \( Y_n = X_n1\{|X_n| < n^{1/r}\} \).
(a) Show that the sequences \( \{X_n\} \) and \( \{Y_n\} \) are Khinchine - equivalent.
(b) What can you say about the sequence \( M_{n,r} = n^{-1/r} \max_{1 \leq k \leq n} |X_k|^r \)?

**Solution:**
(a) Now \( \{X_n \neq Y_n\} = \{|X_n| \geq n^{1/r}\} = \{|X_n|^r \geq n\} \) and hence
\[
\sum_{n=1}^\infty P(X_n \neq Y_n) = \sum_{n=1}^\infty P(|X_n|^r \geq n) = \sum_{n=1}^\infty P(|X_1|^r \geq n) \leq E|X_1|^r < \infty.
\]
Thus the sequences \( \{X_n\} \) and \( \{Y_n\} \) are Khinchine equivalent.
(b) Since \( E|X_1|^r < \infty \) it follows from the SLLN that \( M_{n,r} = n^{-1/r} \max_{1 \leq k \leq n} |X_k|^r \to_a s 0 \).
6. (28 points) Let \(X_1, \ldots, X_n\) be i.i.d. and suppose that \(xP(|X_1| > x) \to 0\) as \(x \to \infty\). For \(k \in \{1, \ldots, n\}\), let \(Y_{k,n} \equiv X_k1_{[|X_k|\leq n]}\).
(a) Show that \(P(\bigcup_{k=1}^n [Y_{k,n} \neq X_k]) \to 0\) as \(n \to \infty\).
(b) State Feller's Weak Law of Large Numbers.

Solution: (a) Now \(P(\bigcup_{k=1}^n [Y_{k,n} \neq X_k]) \leq \sum_{k=1}^n P(Y_{k,n} \neq X_k) = \sum_{k=1}^n P(|X_k| > n) = nP(|X_1| > n) \to 0\) as \(n \to \infty\).
(b) Feller’s weak law of large numbers: If \(X_{n1}, \ldots, X_{nn}\) are i.i.d. for each \(n\), then the following are equivalent:
(i) \(\overline{X}_n - \mu_n \to_p 0\) for some constants \(\mu_n\).
(ii) \(\tau(x) \equiv xP(|X_1| > x) \to 0\) as \(x \to \infty\).
(iii) \(M_n \equiv n^{-1} \max_{1 \leq k \leq n} |X_{n,k}| \to_p 0\).

7. (25 points) Suppose that \(P\) is the measure with density \(p(x) = (1/3)1_{[0,3]}(x)\) with respect to Lebesgue measure \(\lambda\) (on the Borel \(\sigma\)-field of \(\mathbb{R}\)), and \(Q\) is the measure with density \(q(x) = (1/3)1_{[2,5]}(x)\) with respect to Lebesgue measure \(\lambda\).
(a) Is \(P \ll Q\)? Why or why not?
(b) Give the Lebesgue decomposition of \(P\) with respect to \(Q\).
(c) Is \(Q \ll P + Q\)? Why or why not?
(d) Give the Lebesgue decomposition of \(Q\) with respect to \(P + Q\).
(e) If \(\phi \equiv P - Q\), find \(|\phi| (\mathbb{R})\).

Solution: (a) No. Since \(Q([0,2)) = 0\), but \(P([0,2)) = 2/3 > 0\).
(b) The Lebesgue decomposition of \(P\) with respect to \(Q\) is given by
\[
P = P_{ac} + P_s
\]
where
\[
P_{ac}(A) = \int_{A \cap [2,5]} 1 \cdot dQ,
\]
\[
P_s(A) = \int_{A \cap [0,2]} (1/3)dx.
\]
Note that \(P_s([2,5]) = 0\) while \(P_{ac}([2,5]^c) = 0\).
(c) Yes. If \((P + Q)(A) = 0\), then both \(P(A) = 0\) and \(Q(A) = 0\), so in
particular \( Q \ll P + Q \).

(d) We can write

\[
Q(A) = \int_A dQ = \int_A q d\lambda = \int_A \frac{q}{p + q} (p + q) d\lambda = \int_A \frac{q}{p + q} d(P + Q)
\]

where \( q/(p + q) = 1/2 \) for on \([2, 3]\), \( q/(p + q) = 1 \) on \((3, 5]\).

(e) If \( \phi = P - Q \) so that \( \phi(A) = \int_A d(P - Q) = \int_A (p - q) d\lambda \), where

\[
p - q = (1/3)1_{[0, 2]} - (1/3)1_{[3, 5]}, \text{ then } |\phi|(A) = (1/3) \int_A \{1_{[0, 2]} + 1_{[3, 5]}\} d\lambda,
\]

and hence \( |\phi|([0, 1]) = (1/3)(2 + 2) = 4/3 \). (Also note that \( d_{TV}(P, Q) = (1/2) \int |p - q| d\lambda = (1/2)\{ \int_0^2 (1/3) d\lambda + \int_3^5 (1/3) d\lambda = 2/3 \)

which gives agreement via a homework problem.)

8. (25 points). Let \( P_\mu \) denote the distribution of a \( N(\mu, 1) \) random variable \( X \) on \( \mathbb{R} \): thus \( (dP_\mu/d\lambda)(x) = \phi(x - \mu) \) where \( \lambda \) is Lebesgue measure on \( \mathbb{R} \) and \( \phi(x) = (2\pi)^{-1/2} \exp(-x^2/2) \).

(a) Show that \( P_\mu \ll P_0 \).

(b) Find the Radon-Nikodym derivative \( dP_\mu/dP_0 \).

**Solution:**

(a) Suppose that \( P_0(A) = 0 \). But since \( P_0(A) = \int_A \phi(x) d\lambda(x) \)

where \( \phi(x) > 0 \) for all \( x \in \mathbb{R} \), this implies \( \lambda(A) = 0 \). Then

\[
P_\mu(A) = \int_A \phi(x - \mu) d\lambda(x) = 0,
\]

and hence \( P_\mu \ll P_0 \).

(b) Now

\[
P_\mu(A) = \int_A \phi(x - \mu) d\lambda(x) = \int_A \phi(x - \mu) \phi(x) d\lambda(x)
\]

\[
= \int_A \exp(\mu x - \mu^2/2) dP_0(x).
\]

so \( (dP_\mu/dP_0)(x) = \exp(\mu x - \mu^2/2) \).

9. (30 points). (a) Suppose that \( H(x) = \int_{-\infty}^x h(t) dt \) where \( h(t) \geq 0 \) for all \( t \in R \). Use the theorem of the unconscious statistician and Fubini’s theorem to show that

\[
EH(X) = \int_{-\infty}^\infty h(t) P(X \geq t) dt.
\]
(b) Suppose that $X$ is a random variable with values in $[1, \infty)$; i.e. $P(X \geq 1) = 1$. Let $F$ be the distribution function of $X$: $F(x) = P(X \leq x)$. Use the result of (a) (or another direct application of the theorem of the unconscious statistician and Fubini’s theorem) to prove the following formula:

$$E \log X = \int_1^\infty (1 - F(t)) \frac{1}{t} dt.$$

(c) Give an example of a distribution function $F$ of a random variable $X$ such that $E \log(X) < \infty$ but $E(X^r) = \infty$ for all $r > 0$.

**Solution:**

(a) Now

$$EH(X) = \int H(x)dF(x) = \int_{-\infty}^\infty \left( \int_{-\infty}^x h(t)dt \right) dF(x)$$

$$= \int_{-\infty}^\infty \left( \int_{-\infty}^x 1_{[t \leq x]}h(t)dt \right) dF(x)$$

$$= \int_{-\infty}^\infty \left( \int_{-\infty}^\infty 1_{[t \leq x]}dF(x) \right) h(t)dt$$

$$= \int_{-\infty}^\infty P(X \geq t)h(t)dt$$

$$= \int_{-\infty}^\infty h(t)(1 - F(t))dt$$

since $P(X \geq t) = P(X > t)$ a.e. with respect to Lebesgue measure $\lambda$.

(b) When $P(X \geq 1) = 1$ and $H(x) = \log x$, note that

$$H(x) = \log x = \int_1^x \frac{1}{t} dt = \int_{-\infty}^x 1_{[1,\infty)}(t) \frac{1}{t} dt$$

so the hypothesis of (a) holds with $h(t) = 1_{[1,\infty)}(t)t^{-1}$. Thus from (a) it follows that

$$E \log X = \int_{-\infty}^\infty h(t)(1 - F(t))dt = \int_{1}^\infty \frac{1}{t}(1 - F(t))dt.$$

(c) Let $1 - F(t) \equiv (\log(et))^{-\gamma}$ for $t \geq 1$ and for some $\gamma > 1$. Then for $\gamma = 2$

$$E \log X = \int_{1}^\infty \frac{1}{t(\log(et))^2} dt = \int_{1}^\infty y^{-2}dy = 1 < \infty.$$
On the other hand, for $r > 0$

$$EX^r = \int_0^\infty rt^{r-1}(1-F(t))dt = \int_0^1 rt^{r-1}dt + r \int_1^\infty rt^{r-1}\frac{1}{(\log(et))^2}dt$$

$$= 1 + r \int_1^\infty \frac{1}{t^{1-r}(\log(et))^2}dt$$

$$= 1 + r \int_1^\infty \frac{e^{r(y-1)}}{y^2}dy = +\infty$$

10. (30 points) Suppose that $X$ and $Y$ are independent random variables and that $f$ and $g$ are real-valued measurable functions from $(\mathbb{R}, \mathcal{B})$ to $(\mathbb{R}, \mathcal{B})$ such that $f(X)$ and $g(Y)$ are measurable. Suppose that we know that

$$E[f(X)g(Y)] = E[f(X)]E[g(Y)] \quad (1)$$

holds for $f = 1_A$ and $g = 1_B$ for sets $A, B \in \mathcal{B}$.

(i) Show that (1) holds for $f = 1_A$ with $A \in \mathcal{B}$ and a non-negative (measurable) function $g$.

(ii) Using the result of (i), show that (1) holds for $f$ integrable (i.e. $E|f(X)| < \infty$) and $g \geq 0$ (and measurable).

(iii) Using the result of (ii), show that (1) holds for $f$ integrable and $g$ integrable (and measurable).

Solution: (i) Suppose that (1) holds for indicator functions $f = 1_A$ and $g = 1_B$. Fix a measurable set $A$ and let $f = 1_A$. For a given non-negative, measurable function $g$, let $g_n = \sum_{j=1}^m a_j 1_{B_j}$ be a sequence of simple functions with $g_n \nearrow g$. Then, using the Monotone Convergence
Theorem,

\[ E\{1_A(X)g(Y)\} = E\{1_A(X)\lim g_n(Y)\} = \lim_n E\{1_A(X)g_n(Y)\} \]

\[ = \lim_n E\{1_A(X)\sum_{j=1}^m a_j 1_{B_j}(Y)\} = \lim_n \sum_{j=1}^m a_j E\{1_A(X)1_{B_j}(Y)\} \]

\[ = \lim_n \sum_{j=1}^m a_j E\{1_A(X)\} E\{1_{B_j}(Y)\} \quad \text{by our hypothesis} \]

\[ = E\{1_A(X)\} \lim_n \sum_{j=1}^m a_j E\{1_{B_j}(Y)\} \]

\[ = E\{1_A(X)\} \lim_n E\{\sum_{j=1}^m a_j 1_{B_j}(Y)\} = E\{1_A(X)\} E\{g(Y)\} \]

by the Monotone Convergence Theorem again. Thus the formula holds for \( f = 1_A \) and \( g \geq 0 \).

(ii) Now suppose that \( f \geq 0, \ g \geq 0 \) are both measurable. Then there exists a sequence of simple functions \( f_n = \sum_{i=1}^p a_j 1_{A_j} \nearrow f \). Thus by the Monotone Convergence Theorem

\[ E\{f(X)g(Y)\} = E\{\lim_n f_n(X)g(Y)\} = \lim_n E\{f_n(X)g(Y)\} \]

\[ = \lim_n E\{\sum_{i=1}^p a_j 1_{A_i}(X)g(Y)\} = \lim_n \sum_{i=1}^p a_j E\{1_{A_i}(X)g(Y)\} \]

\[ = \lim_n \sum_{i=1}^p a_j E\{1_{A_i}(X)\} E\{g(Y)\} \quad \text{by part (i)} \]

\[ = \lim_n E\{\sum_{i=1}^p a_j 1_{A_i}(X)\} E\{g(Y)\} \quad \text{by linearity of expectation} \]

\[ = \lim_n E\{f_n(X)\} Eg(Y) = Ef(X)Eg(Y) \]

by the Monotone Convergence Theorem again. Thus (1) holds for \( f \geq 0, \ g \geq 0 \). To extend to a general integrable function \( f = f^+ - f^- \) and
\( g \geq 0 \), write

\[
E\{ f(X)g(Y) \} = E\{ (f^+(X) - f^-(X))g(Y) \} = E\{ f^+(X)g(Y) \} - E\{ f^-(X)g(Y) \} \\
= Ef^+(X)Eg(Y) - Ef^-(X)Eg(Y)
\]

by the result just proved for non-negative \( f, g \)

\[
= (Ef^+(X) - Ef^-(X))Eg(Y) = Ef(X)Eg(Y).
\]

Thus the formula holds for a general integrable function \( f \) and \( g \geq 0 \).

(iii) Now let \( f \) and \( g \) be general integrable functions. Then since

\[
g = g^+ - g^-
\]

we can write

\[
E\{ f(X)g(Y) \} = E\{ f(X)(g^+(Y) - g^-(Y)) \} = E\{ f(X)g^+(Y) \} - E\{ f(X)g^-(Y) \} \\
= E\{ f(X) \} Eg^+(Y) - E\{ f(X) \} Eg^-(Y) \\
= E\{ f(X) \} (Eg^+(Y) - Eg^-(Y)) \\
= E\{ f(X) \} E\{ g(Y) \}.
\]

Thus the formula (1) holds for general integrable \( f, g \).

11. (27 points). Let \( X_1, X_2, \ldots \) be i.i.d. with distribution function \( F \) given by

\[
F(x) = \begin{cases} 
\frac{1}{2} \exp(-|x|^4/3), & x \leq 0 \\
1 - \frac{1}{2} \exp(-|x|^4/3), & x \geq 0 
\end{cases}
\]

(2)

and let \( M_n \equiv \max_{1 \leq k \leq n} X_k \).

(a) Show that

\[
\limsup_{n \to \infty} \frac{X_n}{(3 \log n)^{1/4}} = 1 \quad \text{a.s.}
\]

(b) Show that \( M_n/(3 \log n)^{1/4} \to a.s. 1 \).

Solution: (a) Let \( \epsilon > 0 \). Then

\[
P(X_n > (1 + \epsilon)(3 \log n)^{1/4}) = 2^{-1} \exp\left(- (1 + \epsilon)^4 \log n \right) = 2^{-1} n^{-(1+\epsilon)^4},
\]

and hence

\[
\sum_{n=1}^{\infty} P(X_n > (1 + \epsilon)(3 \log n)^{1/4}) = \sum_{n=1}^{\infty} n^{-(1+\epsilon)^4} < \infty.
\]

Since the \( X_n \)'s are independent, this implies (by the first Borel-Cantelli lemma), that

\[
P(X_n > (1 + \epsilon)(3 \log n)^{1/4} \text{ i.o.}) = 0.
\]
This yields
\[
\limsup_{n \to \infty} \frac{X_n}{(3 \log n)^{1/4}} \leq 1 \quad \text{a.s.} \tag{3}
\]
On the other hand, taking \( \epsilon = 0 \) yields \( P(X_n \geq (3 \log n)^{1/4}) = 2^{-1}n^{-1} \), and hence
\[
\sum_{n=1}^{\infty} P(X_n > (3 \log n)^{1/4}) = \sum_{n=1}^{\infty} 2^{-1}n^{-1} = \infty,
\]
so \( P(X_n \geq (3 \log n)^{1/4} \ i.o.) = 1 \) by the second Borel-Cantelli lemma. This yields
\[
\limsup_{n \to \infty} \frac{X_n}{(3 \log n)^{1/4}} \geq 1 \quad \text{a.s.} \tag{4}
\]
Combining (3) and (4) we conclude that
\[
\limsup_{n \to \infty} \frac{X_n}{(3 \log n)^{1/4}} = 1 \quad \text{a.s.} \tag{5}
\]
(b) Note that \( \{M_n\} \) is non-decreasing and \( M_n \geq X_n \). Thus from (a) we see that
\[
\limsup_{n \to \infty} \frac{M_n}{(3 \log n)^{1/4}} \geq \limsup_{n \to \infty} \frac{X_n}{(3 \log n)^{1/4}} = 1 \quad \text{a.s.}
\]
But from (a) we also know that for each \( \epsilon > 0 \) we have \( X_k/(3 \log k)^{1/4} \leq (1 + \epsilon) \) for all \( k \geq \) some \( N_\omega \) for all \( \omega \) in a set with probability 1. Thus
\[
\frac{M_n}{(3 \log n)^{1/4}} \leq \max_{k \leq N_\omega} X_k \frac{1}{(3 \log n)^{1/4}} \leq \max_{k \leq N_\omega} (1 + \epsilon)(3 \log k)^{1/4} \to 0 \vee (1 + \epsilon) = 1 + \epsilon.
\]
Since \( \epsilon > 0 \) was arbitrary, this yields the claim: \( M_n/(3 \log n)^{1/4} \to_{a.s.} 1 \).

12. (27 points) Let \( X_1, X_2, \ldots \) be i.i.d. exponential (1) random variables; i.e. \( P(X_1 > x) = e^{-x} \) for all \( x \geq 0 \). Let \( M_n \equiv \max_{1 \leq k \leq n} X_k \).
(a) Show that \( \limsup_{n \to \infty} (X_n / \log n) = 1 \) a.s.

(b) Show that \( M_n / \log n \to_{a.s.} 1 \).

(c) Show that \( M_n - \log n \to_d \) some \( Y \) and find the distribution function of \( Y \).

**Solution:** (a) For any \( \epsilon > 0 \) we have

\[
P(X_n > (1 + \epsilon) \log n) = e^{-(1+\epsilon) \log n} = n^{-(1+\epsilon)}.
\]

so \( P(X_n > (1 + \epsilon) \log n) \) i.o. = 0 by the first Borel-Cantelli lemma. Similarly, for \( \epsilon \in [0, 1) \),

\[
P(X_n > (1 - \epsilon) \log n) = e^{-(1-\epsilon) \log n} = n^{-(1-\epsilon)},
\]

and hence \( P(X_n > (1 - \epsilon) \log n) \) i.o. = 1 by the second Borel-Cantelli lemma. Thus we conclude that \( \limsup_{n \to \infty} (X_n / \log n) = 1 \) a.s.

(b) Since \( M_n \) is non-decreasing and \( M_n \geq X_n \), it follows that

\[
\limsup_{n \to \infty} \frac{M_n}{\log n} \geq \limsup_{n \to \infty} \frac{X_n}{\log n} = 1 \text{ a.s.}
\]

But from (a), for each \( \epsilon > 0 \) we have \( X_k / \log k \leq (1 + \epsilon) \) for all \( k \geq \) some \( N_\omega \) on a set with probability 1. Thus

\[
\frac{M_n}{\log n} \leq \frac{\max_{k \leq N_\omega} X_k}{\log n} \leq \frac{\max_{k \leq N_\omega} (1 + \epsilon) \log k}{\log n}
\]

\[
\to 0 \vee (1 + \epsilon) = 1 + \epsilon.
\]

Since \( \epsilon > 0 \) was arbitrary, this yields the claim: \( M_n / \log n \to_{a.s.} 1 \).

(c) Now

\[
P(M_n - \log n \leq x) = P(M_n \leq x + \log n) = P(\cap_{k=1}^n [X_k \leq x + \log n])
\]

\[
= (1 - \exp(-(x + \log n)))^n = (1 - n^{-1} e^{-x})^n
\]

\[
\to \exp(-\exp(-x))
\]

for all \( x \in \mathbb{R} \). Thus \( M_n - \log n \to_d Y \) where \( Y \) has the extreme value (or Gumbel) distribution function \( F_Y(y) = \exp(-\exp(-y)) \).