1. PfS, Exercise 1.1.3, page 9:
(a) The minimal \( \lambda \)–system generated by the class \( \mathcal{D} \) is denoted by \( \lambda[\mathcal{D}] \). Show that \( \lambda[\mathcal{D}] \) is equal to the intersection of all \( \lambda \)–systems containing \( \mathcal{D} \).
(b) A collection \( \mathcal{A} \) of subsets of \( \Omega \) is a \( \sigma \)–field if and only if it is both a \( \pi \)–system and a \( \lambda \)–system.
(c) Let \( \mathcal{C} \) be a \( \pi \)–system and let \( \mathcal{D} \) be a \( \lambda \)–system. Then \( \mathcal{C} \subset \mathcal{D} \) implies that \( \sigma[\mathcal{C}] \subset \mathcal{D} \).

**Solution:** (a) Let

\[
\lambda[\mathcal{D}] \equiv \cap \{ \mathcal{F}_\alpha : \mathcal{F}_\alpha \text{ is a } \lambda \text{ – system with } \mathcal{D} \subset \mathcal{F}_\alpha \}.
\]

Now \( \Omega \in \mathcal{F}_\alpha \) for all \( \alpha \), so \( \Omega \in \lambda[\mathcal{D}] \). Further, if \( A, B \in \lambda[\mathcal{D}] \) with \( B \subset A \), then \( B, A \in \mathcal{F}_\alpha \) for all \( \alpha \) and hence \( A \setminus B \in \mathcal{F}_\alpha \) for all \( \alpha \), and hence \( A \setminus B \in \lambda[\mathcal{D}] \). Finally, if \( \{ A_1, A_2, \ldots \} \) is an increasing family of sets in \( \lambda[\mathcal{D}] \), then \( \{ A_1, A_2, \ldots \} \subset \mathcal{F}_\alpha \) for all \( \alpha \). Hence \( \lim_{n} A_n \in \mathcal{F}_\alpha \) for all \( \alpha \), and hence \( \lim_{n} A_n \in \mathcal{F}_\alpha \) for all \( \alpha \). Thus \( \lim_{n} A_n \subset \mathcal{F}_\alpha \) for all \( \alpha \). Thus it follows that \( \lim_{n} A_n \in \lambda[\mathcal{D}] \). Thus \( \lambda[\mathcal{D}] \) is a \( \lambda \)–system. If \( \mathcal{A}' \) is a \( \lambda \)–system such that \( \mathcal{D} \subset \mathcal{A}' \), then \( \mathcal{A} = \mathcal{F}_\alpha \) for some \( \alpha \), and hence \( \lambda[\mathcal{D}] \subset \mathcal{A}' \); i.e., \( \lambda[\mathcal{D}] \) is the minimal \( \lambda \)–system containing \( \mathcal{D} \).

(b) Suppose first that \( \mathcal{A} \) is a \( \sigma \)–field. Thus it is closed under countable intersections, and hence, in particular it is closed under finite intersections and is a \( \pi \)–system. To show that \( \mathcal{A} \) is a \( \lambda \)–system, first consider \( A, B \in \mathcal{A} \) with \( A \subset B \). Since \( \mathcal{A} \) is closed under complementation, \( A^c \in \mathcal{A} \), and hence also \( B \cap A^c = B \setminus A \in \mathcal{A} \). Also \( \Omega \in \mathcal{A} \) since it is a \( \sigma \)–field. Finally, if \( \{ A_n \} \) is a sequence of sets in \( \mathcal{A} \) with \( A_n \uparrow A \), then \( A = \lim_{n} A_n = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A} \) since \( \mathcal{A} \) is a \( \sigma \)–field. Thus \( \mathcal{A} \) is a \( \lambda \)–system, and this completes the proof that a \( \sigma \)–field is both a \( \pi \)–system and a \( \lambda \)–system.

Now suppose that \( \mathcal{A} \) is a \( \pi \)–system and a \( \lambda \)–system. Let \( A \in \mathcal{A} \). Since \( \mathcal{A} \) is a \( \lambda \)–system, \( \Omega \in \mathcal{A} \). Since \( A \subset \Omega \) and \( \mathcal{A} \) is a \( \lambda \)–system,
A^c = \Omega \cap C^c = \Omega \setminus A \in \mathcal{A}. To show that \mathcal{A} is closed under countable unions, suppose that \{A_n\} is a countable family of sets with \( A_n \in \mathcal{A} \) for each \( n \). Set \( B_n \equiv \bigcup_{i=1}^{n} A_i \). Then \( B_n \in \mathcal{A} \) for each \( n \) since \( A, B \in \mathcal{A} \) implies that \( A \cup B = (A^c \setminus B)^c \in \mathcal{A} \) since \( \mathcal{A} \) is a \( \pi \)-system and a \( \lambda \)-system implies that it is closed under intersections, complements, and set differences. But since \( \mathcal{A} \) is a \( \lambda \)-system this implies that \( \bigcap_{n=1}^{\infty} B_n = \lim_{n} \bigcup_{i=1}^{n} A_i = \lim_{n} \bigcup_{i=1}^{n} B_i = \lim_{n} B_n \in \mathcal{A} \), and hence \( \mathcal{A} \) is closed under countable unions. Hence \( \mathcal{A} \) is a \( \sigma \)-field.

(c) It is clear that \( \lambda(C) \subset \sigma[C] \) (since there are fewer restrictions in defining a \( \lambda \)-system than a field; or from (b)). If we show that \( \lambda[C] \) is a \( \pi \)-system, then since \( \sigma[C] \) is a \( \lambda \)-system containing \( C \), it must also be the minimal \( \lambda \)-system containing \( C \) and hence \( \sigma[C] = \lambda[C] \subset \lambda[D] \subset \mathcal{D} \). Thus it suffices to show that \( \lambda[C] \) is a \( \pi \)-system.

We do this in two steps:

**Step 1:** Let

\[ \mathcal{D}_1 = \{ B \in \lambda(C) : B \cap C \in \lambda(C) \text{ for all } C \in \mathcal{C} \} \]

where \( \mathcal{C} \) is a \( \pi \)-system and \( \lambda(C) \) is the smallest \( \lambda \)-system containing \( C \). To show that \( \mathcal{D}_1 \) is a \( \lambda \)-system we need to show that:

(i) \( \Omega \in \mathcal{D}_1 \).

(ii) If \( D_n \in \mathcal{D}_1 \), \( D_n \uparrow \), then \( \bigcup D_n \in \mathcal{D}_1 \).

(iii) If \( A, B \in \mathcal{D}_1 \) with \( A \subset B \), then \( A \setminus B \in \mathcal{D}_1 \).

Proof of (i): \( \Omega \in \lambda(C) \) since it is a \( \lambda \)-system, so we have \( \Omega \cap C = C \in \mathcal{C} \subset \lambda(C) \) for each \( C \in \mathcal{C} \), and hence \( \Omega \in \mathcal{D}_1 \).

Proof of (ii): Suppose \( D_1, D_2, \ldots \in \mathcal{D}_1 \) and \( D_n \uparrow \). Then we have \( \bigcup_{n} D_n \in \lambda(\mathcal{D}) \) (since each \( D_n \in \lambda(\mathcal{D}) \), a \( \lambda \)-system), and \( (\bigcup_{n} D_n) \cap C = \bigcup_{n}(D_n \cap C) = \bigcup_{n} B_n \in \lambda(C) \) since \( B_n \in \lambda(C) \) is \( \uparrow \). Hence \( \bigcup_{n} D_n \in \mathcal{D}_1 \).

Proof of (iii): Suppose \( A, B \in \mathcal{D}_1 \) with \( B \subset A \). Then \( AC, BC, A, B \in \lambda(C) \) for all \( C \in \mathcal{C} \), so

\[ (A \setminus B) \cap C = (A \cap C) \setminus (B \cap C) \in \lambda(C) \]

for all \( C \in \mathcal{C} \). Hence \( A \setminus B \in \mathcal{D}_1 \).

**Step 2:** Let

\[ \mathcal{D}_2 \equiv \{ A \in \lambda[C] : B \cap A \in \lambda[C], \text{ for all } B \in \lambda[C] \} \]
Step 1 showed that $D_2$ contains $C$. As in step 1, we can show that $D_2$ inherits the $\lambda-$system structure form $\lambda[C]$ and that therefore $D_2 = \lambda[C]$. But the fact that $\lambda[C] = D_2$ means that $\lambda[C]$ is a $\pi-$system.

2. PfS, Exercise 1.2.1, page 15. Let $(\Omega, \mathcal{A}, \mu)$ denote a measure space. Show that

$$
\hat{\lambda}_\mu \equiv \{ A : A_1 \subset A \subset A_2, A_1, A_2 \in \mathcal{A}, \mu(A_2 \setminus A_1) = 0 \}
$$

$$
= \{ A \cup N : A \in \mathcal{A} \text{ and } N \subset (\text{some } B) \in \mathcal{A} \text{ with } \mu(B) = 0 \}
$$

$$
= \{ A \triangle N : A \in \mathcal{A}, N \subset (\text{some } B) \in \mathcal{A} \text{ with } \mu(B) = 0 \}
$$

and is a $\sigma-$field. Show that $(\Omega, \hat{\lambda}_\mu, \hat{\mu})$ is complete.

**Solution:** Let these three collections of sets be called $\hat{\mathcal{A}}_1$, $\hat{\mathcal{A}}_2$, and $\hat{\mathcal{A}}_3$ respectively. To show that $\hat{\mathcal{A}}_1 \subset \hat{\mathcal{A}}_2$, suppose that $D \in \hat{\mathcal{A}}_1$. Then there exist sets $A_1, A_2 \in \mathcal{A}$ such that $A_1 \subset D \subset A_2$ and $\mu(A_2 \setminus A_1) = 0$. Let $B = A_2 \setminus A_1$; then $\mu(B) = 0$ and $B \in \mathcal{A}$. Furthermore $D = A_1 \cup N$ for some $N \subset B$. Hence with $A = A_1$ and $B = A_2 \setminus A_1$ we have $D = A \cup N$ where $A \in \mathcal{A}$ and $N \subset B$ with $\mu(B) = 0$. Thus $D \in \hat{\mathcal{A}}_2$, and $\hat{\mathcal{A}}_1 \subset \hat{\mathcal{A}}_2$.

To show that $\hat{\mathcal{A}}_2 \subset \hat{\mathcal{A}}_3$, let $D \in \hat{\mathcal{A}}_2$. Then $D = A \cup N$ with $A \in \mathcal{A}$, $N \subset B \in \mathcal{A}$ having $\mu(B) = 0$, so $A \subset D$. Let $N_1 \equiv D \setminus A = D \cap A^c = N \cap A^c \subset N \subset B$. Hence we have

$$
A \triangle N_1 = (A^c \cap (D \cap A^c)) \cup (A \cap (D \cap A^c)^c)
$$

$$
= (D \cap A^c) \cup ((A \cap D^c) \cup A)
$$

$$
= (D \cap A^c) \cup A = D \cup A = D
$$

and hence $D \in \hat{\mathcal{A}}_3$.

To show that $\hat{\mathcal{A}}_3 \subset \hat{\mathcal{A}}_1$, let $D \in \hat{\mathcal{A}}_3$. Then $D = A \triangle N$ with $A \in \mathcal{A}$ and $N \subset B \in \mathcal{A}$ with $\mu(B) = 0$. Take $A_1 = A \cap B^c$, $A_2 = A \cup B$. Then $A_1 \subset A \cap N^c \subset D \subset A \cup N \subset A_2$. Since $A_2 \setminus A_2 = (A \cup B) \setminus (A \cap B^c)^c = A \cap (A^c \cup B) \cup B \cap A^c \cap B = (A \cap B) \cup (A^c \cap B) = B$ so that $\mu(A_2 \setminus A_1) = \mu(B) = 0$, it follows that $D \in \hat{\mathcal{A}}_1$, and hence that $\hat{\mathcal{A}}_3 \subset \hat{\mathcal{A}}_1$.

Since we have shown that

$$
\hat{\mathcal{A}}_1 \subset \hat{\mathcal{A}}_2 \subset \hat{\mathcal{A}}_3 \subset \hat{\mathcal{A}}_1,
$$

3
it follows that $\hat{A}_1 = \hat{A}_2 = \hat{A}_3$.

To show that $\hat{A}_\mu$ is a $\sigma$-field:

Let $A \in \hat{A}_1$. Then $A_1 \subset A \subset A_2$, so that $A_2^c \subset A^c \subset A_1^c$ with $A_2^c, A_1^c \in \mathcal{A}$, and where $A_1^c \setminus A_2^c = A_2 \setminus A_1$ and hence $\mu(A_1^c \setminus A_2^c) = \mu(A_2 \setminus A_1) = 0$. Hence $A^c \in \hat{A}_1 = \hat{A}_\mu$.

Let $D_1, \ldots, D_n \in \hat{A}_2$. Then $D_n = A_n \cup N_n$ with $A_n \in \mathcal{A}$, $N_n \subset B_n \in \mathcal{A}$ with $\mu(B_n) = 0$ for each $n$. Thus $\cup A_n \in \mathcal{A}$, $\cup B_n \in \mathcal{A}$, $\cup N_n \subset \cup B_n$, and $\mu(\cup B_n) \leq \sum \mu(B_n) = 0$. Hence $\cup D_n = (\cup A_n) \cup (\cup N_n) \in \hat{A}_2$. Hence $\hat{A}_\mu$ is a $\sigma$-field.

To show that $(\Omega, \hat{A}_\mu, \hat{\mu})$ is complete, first let $A_1 \cup N_1 = A_2 \cup N_2$ with $A_1, A_2 \in \mathcal{A}$, $N_1 \subset B_1$, $N_2 \subset B_2$ with $\mu(B_1) = \mu(B_2) = 0$. By definition $\hat{\mu}(A_1 \cup N_1) = \mu(A_1)$ $\hat{\mu}(A_2 \cup N_2) = \mu(A_2)$. But $A_1 \subset A_1 \cup N_1 = A_2 \cup N_2 \subset A_2 \cup B_2$ and similarly $A_2 \subset A_1 \cup B_1$. Hence we have $\mu(A_1) \leq \mu(A_2) + \mu(B_2) = \mu(A_2)$ and $\mu(A_2) \leq \mu(A_1) + \mu(B_1) = \mu(A_1)$, or $\mu(A_1) = \mu(A_2)$. Thus $\hat{\mu}$ is well-defined. That it extends $\mu$ is trivial.

To show completeness, let $D \subset (\text{some } B) \in \hat{A}_\mu$ with $\mu(B) = 0$. Then $D = \emptyset \cup D \in \hat{A}_2 = \hat{A}_\mu$.

3. PfS, Exercise 1.2.3, page 16: Suppose that $\mu$ on a field $\mathcal{C}$ is $\sigma$–finite on $\mathcal{C}$ and is extended to $\mathcal{A} = \sigma(\mathcal{C})$; call the extension $\mu$.

(a) For each $A \in \mathcal{A}$ with $\mu(A) < \infty$ and each $\epsilon > 0$ there exists a set $C = C_\epsilon \in \mathcal{C}$ such that $\mu(A \triangle C) < \epsilon$.

(b) Let $\mu$ denote counting measure on the integers. Then $\mathcal{C} = \{C : C$ or $C^c$ is finite$\}$ is a field. Determine $\sigma[\mathcal{C}]$. Show that the conclusion of part (a) fails for the set of even integers.

**Solution:** Proof of (a): Now

$$\mu(A) = \inf \\{ \sum_n \mu(A_n) : A \subset \cup_{n=1}^\infty A_n \text{ with all } A_n \in \mathcal{C} \}.$$  

Hence there exists $\{A_n\} \subset \mathcal{C}$ such that

$$\sum_{n=1}^\infty \mu(A_n) \leq \mu(A) + \epsilon/2.$$  

Without loss of generality, we may assume that the sets $A_n$ are disjoint (if not, form the disjoint sets $B_1 = A_1, B_n = A_n^c \cap \ldots \cap A_{n-1}^c \cap A_n$,
\(n = 2, 3, \ldots\). Furthermore, there exists an \(N = N_\epsilon\) sufficiently large such that
\[
\sum_{n=N+1}^\infty \mu(A_n) \leq \epsilon/2,
\]
and hence
\[
\sum_{n=1}^\infty \mu(A_n) < \sum_{n=1}^N \mu(A_n) + \epsilon/2 = \mu(\sum_{n=1}^N A_n) + \epsilon/2.
\]
Then \(C \equiv \sum_{n=1}^N A_n \in \mathcal{C},\)
\[
\mu(A \setminus C) \leq \mu(\sum_{n=1}^\infty A_n \setminus C) = \mu(\sum_{n=1}^\infty A_n) - \mu(C) < \epsilon/2
\]
by the choice of \(N,\) and
\[
\mu(C \setminus A) \leq \mu(\sum_{n=1}^\infty A_n \setminus A) = \mu(\sum_{n=1}^\infty A_n) - \mu(A) < \epsilon/2
\]
by the choice of \(\{A_n\}.\) Putting these together gives
\[
\mu(A \triangle C) = \mu(A \setminus C) + \mu(C \setminus A) < \epsilon/2 + \epsilon/2 = \epsilon.
\]
(b) Note that all the singletons \(D_k = \{k\}\) are in \(\mathcal{C},\) and since all subsets of \(\mathbb{Z}\) are either finite or countable, every subset \(A\) of \(\mathbb{Z}\) can be written as a countable union of the singletons \(D_k, k \in A.\) Thus \(\sigma[\mathcal{C}] = 2^\mathbb{Z}.\)
Consider \(A = \{2, 4, 6, \ldots\} = \bigcup_k \{2k\} \equiv \bigcup_k C_k\) where each \(C_k \in \mathcal{C}\) since \(C_k\) itself is a finite set. Note that \(A \notin \mathcal{C}\) since neither \(A\) nor \(A^c = \{1, 3, \ldots\}\) is finite. Thus \(\mu(A) = \infty\) (so the hypothesis of (a) fails). Furthermore, for any set \(C \in \mathcal{C}\)
\[
A \Delta C = (A \cap C^c) \cup (A^c \cap C)
\]
where both \(A\) and \(A^c\) are non-finite sets, and either \(C\) or \(C^c\) is non-finite, and hence at least one of \(A \cap C^c\) and \(A^c \cap C\) is also non-finite. Thus \(\mu(A \Delta C) = \infty\) for all \(C \in \mathcal{C}.\) Hence the conclusion of (a) fails to hold.

4. PfS, Exercise 1.2.4, page 16. Let \(\Omega\) consist of the sixteen values 1, \ldots, 16. (Think of them arranged in four rows of four values.) Let
\[
C_1 = \{1, 2, 3, 4, 5, 6, 7, 8\},
C_2 = \{9, 10, 11, 12, 13, 14, 15, 16\},
C_3 = \{1, 2, 5, 6, 9, 10, 13, 14\},
C_4 = \{3, 4, 7, 8, 11, 12, 15, 16\}.
\]
Let $C = \{C_1, C_2, C_3, C_4\}$, and let $\mathcal{A} = \sigma[C]$.

(a) Show that $\mathcal{A} \equiv \sigma[C] \neq 2^\Omega$.

Proof. Write $\{1, \ldots, 16\}$ in four rows of four numbers each as follows:

<table>
<thead>
<tr>
<th></th>
<th>$C_3$</th>
<th>$C_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$C_2$</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>14</td>
</tr>
</tbody>
</table>

Let $B \equiv \{C_i \cap C_j : i, j \in \{1, \ldots, 4\}\}$. Then it is clear that $(\sigma[C]) = 2^B = 2^4 \neq 2^{10} = (2^\Omega)$. Thus $\sigma[C] \neq 2^\Omega$.

(b) Let $\mu(C_i) = 1/2$ where $1 \leq i \leq 4$ with $\mu(C_1C_3) = 1/4$. Show that $\mathcal{A}_\mu = \mathcal{A}$ with $2^4$ sets.

Proof. Let $B_1 = C_1C_3$, $B_2 = C_1C_4$, $B_3 = C_2C_3$, and $B_4 = C_2C_4$. Then $\mu(B_1) = \mu(C_1C_3) = 1/4$ implies that $\mu(B_i) = 1/4$ for $i = 2, 3, 4$. This holds since $1/2 = \mu(C_1) = \mu(B_1 + B_2) = \mu(B_1) + \mu(B_2) = 1/4 + \mu(B_2)$, so that $\mu(B_2) = 1/4$, and similarly $1/2 = \mu(C_3) = \mu(B_1 + B_3) = \mu(B_1) + \mu(B_3) = 1/4 + \mu(B_3)$, so $\mu(B_3) = 1/4$, and $1/2 = \mu(C_4) = \mu(B_2 + B_4) = \mu(B_2) + \mu(B_4) = 1/4 + \mu(B_4)$, so $\mu(B_4) = 1/4$. Thus the only set $B \in \mathcal{A}$ with $\mu(B) = 0$ is $B = \emptyset$, and it follows that $\mathcal{A}_\mu = \mathcal{A}$ with $\#(\mathcal{A}_\mu) = \#(\mathcal{A}) = 2^4$.

(c) Now suppose $\mu(C_i) = 1/2$ for $i = 2, 3, 4$, but $\mu(C_2C_4) = 0$. Show that $\mathcal{A}_\mu$ contains $2^{10} = 1024$ sets.

Proof. In this case $\mu(B_4) = \mu(C_2C_4) = 0$, and this implies that $\mu(B_2) = 1/2 = \mu(B_3)$ (since $\mu(C_2) = 1/2 = \mu(C_4)$). Thus we also have $\mu(B_1) = 0$ (since $\mu(B_1) + \mu(B_2) = 1/2$). Therefore we need to consider all the sets $N \subset 2^{B_1} + 2^{B_4}$ in forming the completion; that is we need to consider all the sets $\{\{1\}, \{2\}, \{5\}, \{6\}, \{11\}, \{12\}, \{14\}, \{16\}\}$ in forming the completion together with the two basic sets with non-zero probability, $B_2$ and $B_4$. Thus $\#(\mathcal{A}_\mu) = 2^{10} = 1024$.

(d) Illustrate Proposition 1.2.1 in the context of this exercise.

Proof. Consider $\mu$ as given in part (b), and let $B = \{1\}$. Consider extending $\mu$ to $\sigma[\mathcal{A}_\mu \cup \{B\}]$ by defining $\mu(B) = a$ where $0 < a < 1/4$. 

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This is valid extension of $\mu$ for each $a \in (0, 1/4)$, but it is not unique since there are (uncountably) many choices for $a$.

5. PfS, Exercise A.1.5, page 428: It is shown on page 428 that for $W_m \equiv \inf\{t > 0 : N(t) = m\}$ where $N$ is a standard Poisson process on $\mathbb{R}^+$ with intensity $\nu$, we have

$$1 - F_{W_m}(t) = P(W_m > t) = \sum_{k=0}^{m-1} (\nu t)^k e^{-\nu t} / k!.$$ 

By differentiating both sides of this identity show that $W_m$ has density

$$f_{W_m}(t) = \nu (\nu t)^{m-1} e^{-\nu t} / \Gamma(m) \text{ for } t \geq 0.$$

**Solution:** Easy differentiation:

$$-f_{W_n}(t) = \frac{d}{dt} P(W_n > t)$$

$$= \sum_{k=0}^{m-1} \left\{ k(\nu t)^{k-1} \frac{\nu e^{-\nu t}}{k!} - \frac{(\nu t)^k}{k!} \nu e^{-\nu t} \right\}$$

$$= \nu e^{-\nu t} \left\{ \sum_{k=1}^{m-1} \left[ \frac{(\nu t)^{k-1}}{(k-1)!} - \frac{(\nu t)^k}{k!} \right] - 1 \right\}$$

$$= \nu e^{-\nu t} \left\{ \left( 1 - \frac{\nu t}{1} \right) + \left( \frac{\nu t}{1} - \frac{(\nu t)^2}{2!} \right) + \cdots + \left( \frac{(\nu t)^{m-2}}{(m-2)!} - \frac{(\nu t)^{m-1}}{(m-1)!} \right) - 1 \right\}$$

$$= -\nu e^{-\nu t} \frac{(\nu t)^{m-1}}{(m-1)!}.$$

Thus (19) holds.