1. PfS, Exercise 2.2.1, page 28:
Suppose that $\Omega, A = (R_2, B_2)$ where $B_2$ denotes the $\sigma-$field generated by all open subsets of the plane. Recall that this $\sigma-$field contains all sets of the form $B \times R$ and $R \times B$ for all $B \in B$ where $B_1 \times B_2 = \{(r_1, r_2) : r_1 \in B_1, r_2 \in B_2\}$. Now define measurable transformations $X_1(r_1, r_2) = r_1$ and $X_2(r_1, r_2) = r_2$. Then define $Z_1 \equiv \sqrt{X_1^2 + X_2^2}$ and $Z_2 \equiv \text{sign}(X_1 - X_2)$ where $\text{sign}(r) = 1, 0, -1$ according as $r$ is $> 0, = 0, < 0$. Give geometric descriptions of the $\sigma-$fields $F(Z_1)$, $F(Z_2)$, and $F(Z_1, Z_2)$.

**Solution:** The $\sigma-$field $F(Z_1)$ is determined by circles about the origin: if $Z_1$ is known, then we know that $X_1$ and $X_2$ are on a circle with radius $Z_1$. The $\sigma-$field $F(Z_2)$ is the finite $\sigma-$field generated by the three sets $L^+ \equiv \{(r_1, r_2) \in R^2 : r_1 < r_2\}$, $L \equiv \{(r_1, r_2) \in R^2 : r_1 = r_2\}$, and $L^- \equiv \{(r_1, r_2) \in R^2 : r_1 > r_2\}$. Thus if we know $Z_2$, then we know that $(X_1, X_2)$ is either above the forty-five degree line, on this line, or below it. The $\sigma-$field $F(Z_1, Z_2)$ is determined by both the circles generating $F(Z_1)$ and the three sets generating $F(Z_2)$: if we know both $Z_1$ and $Z_2$, then we know that $(X_1, X_2)$ is either on a half-circle of radius $Z_1$ above the diagonal, on the half-circle of radius $Z_1$ where it is intersected by the diagonal, or on the half-circle of radius $Z_1$ and below the diagonal.

2. PfS, Exercise 2.2.2, page 28:
Suppose that $C$ is a $\pi-$system. Suppose that $V$ is a vector space of functions with:
(i) $1_C \in V$ for all $C \in \mathcal{C}$.
(ii) If $A_n \in V$ satisfy $A_n \nearrow A$, then $A \in V$.
(a) Show that $1_A \in V$ for every $A \in \sigma[C]$.
(b) Show that every simple function
$$\sum_{i=1}^{m} x_i 1_{A_i} \text{ is in } V$$
whenever \( m \geq 1 \), \( x_i \in R \), and \( \sum_{1}^{m} A_i = \Omega \) with \( A_i \in \sigma[C] \).
(c) Show that \( \mathcal{V} \) contains all \( \sigma[C] \)-measurable functions.

Solution: (a) Consider the collection of sets \( \mathcal{A} = \{ A \subset \Omega : 1_A \in \mathcal{V} \} \).
For \( C \in \mathcal{C} \) we have \( 1_C \in \mathcal{V} \), by hypothesis, so \( C \in \mathcal{A} \), and hence \( \mathcal{C} \subset \mathcal{A} \).
We will show that \( \mathcal{A} \) is a \( \lambda \)-system:
(1) First note that \( \Omega \in \mathcal{A} \) since \( \Omega \in \mathcal{C} \).
(2) Now suppose that \( A_n \in \mathcal{A} \) \( \nearrow \) \( A \). But then \( 1_{A_n} \in \mathcal{V} \) with \( 1_{A_n} \nearrow 1_A \in \mathcal{V} \) by hypothesis, so \( A \in \mathcal{A} \).
(3) Finally, suppose that \( A, B \in \mathcal{A} \) with \( A \subset B \). Then \( 1_A, 1_B \in \mathcal{V} \) and \( 1_B \setminus A = 1_B - 1_A \in \mathcal{V} \) since \( \mathcal{V} \) is a vector space, and hence \( B \setminus A \in \mathcal{A} \).
Thus \( \mathcal{A} \) is a \( \lambda \)-system and \( \mathcal{C} \subset \mathcal{A} \). Therefore by the \( \pi - \lambda \) theorem, \( \sigma[C] \subset \mathcal{A} \). It follows that \( 1_A \in \mathcal{V} \) for all \( A \in \sigma[C] \).
(b) Since \( \mathcal{V} \) is a vector space, it follows that all simple functions of the form \( \sum_{1}^{m} x_i 1_{A_i} \) with \( x_i \in R \) and \( A_i \in \sigma[C] \), \( i = 1, \ldots, m \) are in \( \mathcal{V} \).
(c) Now suppose that \( X = X^+ - X^- \) is a \( \sigma[C] \)-measurable function.
Since all non-negative \( \sigma[C] \) measurable functions are monotone limits of simple functions and \( \mathcal{V} \) is closed under monotone limits, we conclude that \( X^+, X^- \in \mathcal{V} \), and since \( \mathcal{V} \) is a vector space, this yields \( X \in \mathcal{V} \).

3. PfS, Exercise 2.3.1, page 29:
Let \( X_1, X_2, \ldots \) denote measurable functions from \( (\Omega, \mathcal{A}, \mu) \) to \( (R, \mathcal{B}) \).
(a) If \( X_n \to_{a.e.} X \), then \( X = \bar{X} \) a.e.
(b) If \( X_n \to_{a.e.} X \) and \( \mu \) is complete, then \( X \) itself is measurable.

Solution: (a) Since \( X_n \to_{a.e.} X \), there is a set \( N \in \mathcal{A} \) with \( \mu(N) = 0 \) and \( X_n(\omega) \to X(\omega) \) for all \( \omega \in N^c \). Define \( Y_n = X_n 1_{N^c} \). Then the \( Y_n \)'s are measurable and \( Y_n(\omega) = X_n(\omega)1_{N^c}(\omega) \to X(\omega)1_{N^c}(\omega) \equiv \bar{X} \) for all \( \omega \in \Omega \). Since the \( Y_n \)'s are measurable and converge everywhere to \( \bar{X} \), the limit \( \bar{X} \) is measurable. Furthermore, \( \bar{X}(\omega) = X(\omega) \) for all \( \omega \in N^c \), so \( \bar{X} = X \) a.e.
(b) From part (a) we have \( [\bar{X} \neq X] \subset N \). Since \( \mu \) is complete and \( \mu(N) = 0 \), it follows that \( [\bar{X} \neq X] \in \mathcal{A} \) and \( \mu([\bar{X} \neq X]) = 0 \). Now for any set \( B \in \mathcal{B} \) we can write
\[
X^{-1}(B) = (X^{-1}(B) \cap [X = \bar{X}]) \cup (X^{-1}(B) \cap [X \neq \bar{X}])
= (\bar{X}^{-1}(B) \cap [X = \bar{X}]) \cup C
\]
where \( C = X^{-1}(B) \cap [X \neq \tilde{X}] \subset [X \neq \tilde{X}] \in A \) with \( \mu([X \neq \tilde{X}]) = 0 \). By completeness of \( \mu \) this yields \( C \in A \) and \( \mu(C) = 0 \). But \( X'(B) \in A \) since \( \tilde{X} \) is measurable, and \( [X = \tilde{X}] \in A \), and hence we conclude that \( X^{-1}(B) \in A \). Thus \( X \) is measurable.

4. PfS, Exercise 2.3.2, page 31:
(a) Show that in general \( \rightarrow \mu \) does not imply \( \rightarrow_{a.e.} \).
(b) Give an example with \( \mu(\Omega) = \infty \) where \( \rightarrow_{a.e.} \) does not imply \( \rightarrow \mu \).

Solution: (a) Let \( \Omega = [0, 1] \), and \( \mu = \lambda \) = Lebesgue measure on \([0, 1]\).
Now let \( A_1 = [0, 1/2) \), \( A_2 = [1/2, 1) \), \( A_3 = [0, 1/3), A_4 = [1/3, 2/3) \), \( A_5 = [2/3, 1] \), \ldots . Now let \( X_n(\omega) = 1_{A_n}(\omega) \) for \( n = 1, 2, \ldots \), and let \( X(\omega) = 0 \). Now \( X_n \rightarrow \mu X = 0 \) if \( \mu([|X_n| > \varepsilon]) \rightarrow 0 \) as \( n \rightarrow \infty \) for every \( \varepsilon > 0 \). In this case, for each \( \varepsilon \in (0, 1) \) \( \mu([|X_n| > \varepsilon]) = \mu(A_n) \rightarrow 0 \) as \( n \rightarrow \infty \), so \( X_n \rightarrow_a.e. X = 0 \). However, \( X_n \rightarrow_{a.e.} X = 0 \) iff \( \mu([|X_n| > \varepsilon] \text{ i.o.}) = 0 \) for every \( \varepsilon > 0 \). But for any \( \varepsilon \in (0, 1) \) we have \( \{\omega \in \Omega : |X_n(\omega)| > \varepsilon \ \text{i.o.}\} = [0, 1] \) by construction of the intervals \( A_n \), and hence \( \mu([|X_n| > \varepsilon] \text{ i.o.}) = 1 \). Hence \( X_n \not\rightarrow_{a.e.} X \).
(b) Let \( \Omega = [0, \infty) \) with \( \mu = \lambda \) = Lebesgue measure. Set \( X_n(\omega) = 1_{[n,n+1)}(\omega) \) for \( n = 1, 2, \ldots \), and \( X(\omega) = 0 \). Now \( X_n \not\rightarrow_{a.e.} X \) and in fact, since \( X_n(\omega) = 0 \) for all \( n > \omega \), \( X_n(\omega) \rightarrow X(\omega) = 0 \) for every \( \omega \in \Omega \). But \( X_n \not\rightarrow \mu 0 \) because, for each \( \varepsilon \in (0, 1) \),
\[
\mu([|X_n| > \varepsilon]) = 1 \not\rightarrow 0.
\]

5. PfS, Exercise 2.3.3, page 32. Show that \( X_n \rightarrow \mu X \) if and only if \( X_n - X_m \rightarrow \mu 0 \).

Solution: First suppose that \( X_n \rightarrow \mu X \). Let \( \varepsilon > 0 \). Then we can choose \( N = N_\varepsilon \) so large that for \( n > N_\varepsilon \) we have \( \mu([|X_n - X| > \varepsilon/2]) \leq \varepsilon/2 \).

But then the triangle inequality yields
\[
[\varepsilon < |X_m - X_n| \leq |X_m - X| + |X - X_n|]
\subset [|X_m - X| > \varepsilon/2] \cup [|X_n - X| > \varepsilon/2],
\]
and hence
\[
\mu([|X_m - X_n| > \varepsilon])
\leq \mu([|X_m - X| > \varepsilon/2]) + \mu([|X_n - X| > \varepsilon/2]) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]
Thus \( \{X_n\} \) is Cauchy in measure.

Now suppose that \( X_n - X_m \to \mu \) 0. First, choose a subsequence \( n_k \) increasing so that
\[
\mu([|X_{n_k} - X_l| > 2^{-k}]) < 2^{-k} \quad \text{for all } l > n_k.
\]

Let \( A_k \equiv [|X_{n_k} - X_{n_{k+1}}| > 2^{-k}] \). Set \( B_m \equiv \bigcup_{k=m}^{\infty} A_k \), and note that
\[
\mu(B_m) \leq \sum_{k=m}^{\infty} \mu(A_k) < \sum_{m}^{\infty} 2^{-k} = 2^{-(m-1)}.
\]

On \( B_m^c = \cap_{k=m}^{\infty} A_k^c \) we have \( |X_{n_k} - X_{n_{k+1}}| \leq 2^{-k} \) for all \( k \geq m \). Moreover, for \( n_i > n_j > m \) it follows that
\[
|X_{n_i}(\omega) - X_{n_j}(\omega)| \leq \sum_{k=j}^{\infty} |X_{n_k}(\omega) - X_{n_{k+1}}(\omega)| < 2^{-(j-1)}
\]
for \( \omega \in B_m^c \), and this implies that \( X_{n_k}(\omega) \to X(\omega) \) for all \( \omega \in C \equiv \bigcup_1^{\infty} B_m^c \) with
\[
\mu(C^c) = \mu(\cap_1^{\infty} B_m) \leq \lim \sup \mu(B_m) \leq \lim 2^{-(m-1)} = 0.
\]

Define \( X(\omega) = 0 \) for \( \omega \in C^c \); then \( X \) is measurable, and we have
\[
\mu([|X_n - X| \geq \epsilon]) \leq \mu([|X_n - X_{n_k}| \geq \epsilon/2]) + \mu([|X_{n_k} - X| \geq \epsilon/2]) \to 0
\]
as \( n \geq n_k \to \infty \). Thus \( X_n \to \mu X \).