Due: Wednesday, December 7, 2016

1. PfS, Exercise 7.1.1, page 124:
   (a) Show that \( P(AB) = P(A)P(B) \) if and only if \( \{\emptyset, A, A^c, \Omega\} \) and \( \{\emptyset, B, B^c, \Omega\} \) are independent \( \sigma \)-fields.
   (b) Show that \( A_1, \ldots, A_n \) are independent if and only if (for each \( k = 1, \ldots, n \),
   \[ P(A_{i_1} \cap \cdots \cap A_{i_k}) = \prod_{j=1}^{k} P(A_{i_j}) \quad \text{whenever } 1 \leq i_1 < \cdots < i_k \leq n. \]

Solution: We will prove (b) first.
(b) Since \( A_1, \ldots, A_n \) are independent if and only if the random variables \( 1_{A_1}, \ldots, 1_{A_n} \) are independent, if and only if the \( \sigma \)-fields \( F(1_{A_1}), \ldots, F(1_{A_n}) \) are independent, and of course these are just the \( \sigma \)-fields
\( \mathcal{A}_1 \equiv \{\emptyset, A_1, A_1^c, \Omega\}, \ldots, \mathcal{A}_n \equiv \{\emptyset, A_n, A_n^c, \Omega\} \).

It remains only to show that the \( \sigma \)-fields \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) are independent if and only if for each \( k = 1, \ldots, n \)
\[ P(A_{i_1} \cap \cdots \cap A_{i_k}) = \prod_{j=1}^{k} P(A_{i_j}) \quad \text{(1)} \]
whenever \( 1 \leq i_1 < \cdots < i_k \leq n \). First suppose that the \( \sigma \)-fields \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) are independent. Then by taking \( \Omega \in \{\emptyset, A_j, A_j^c, \Omega\} \) for \( j \in \{i_1, \ldots, i_k\}^c \), we have, with \( B_j = A_{i_m} \) if \( j = i_m \), \( B_j = \Omega \) if \( j \in \{i_1, \ldots, i_k\}^c \),
\[ P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(\cap_{j=1}^{n} B_j) = \prod_{j=1}^{n} P(B_j) = P(A_{i_1}) \cdots P(A_{i_k}) \]
since \( P(B_j) = P(\Omega) = 1 \) for \( j \in \{i_1, \ldots, i_k\}^c \); i.e. (1) holds. Now suppose that (1) holds. In particular, this implies that
\[ P(B_1 \cap \cdots \cap B_n) = P(B_1) \cdots P(B_n) \quad \text{(2)} \]
where each \( B_i \in \{ \emptyset, A_i, \Omega \} \), \( i = 1, \ldots, n \), since both sides are 0 if any \( B_j = \emptyset \), and if \( B_j = A_j \) for \( j \in \{ j_1, \ldots, j_m \} \subset \{ 1, \ldots, n \} \), \( B_j = \Omega \) for \( j \in \{ j_1, \ldots, j_m \}^c \subset \{ 1, \ldots, n \} \), then (2) reduces to (1). Now consider replacing \( B_1 \) by the other remaining element of \( A_i \), \( A_i^c \); if we replace \( B_1 \) by \( A_i^c \), then since \( \Omega = A_1 + A_i^c \) it follows that \( \Omega \cap \cap_{k=2}^n A_k = \cap_{k=1}^n A_k + A_i^c \cap_{k=2}^n A_k \) and hence the left side of (2) becomes

\[
P(\cap_{k=2}^n B_k) - P(A_1 \cap \cap_{k=2}^n B_k) = P(B_2) \cdots P(B_n) - P(A_1)P(B_2) \cdots P(B_n)
\]

\[
= (1 - P(A_1))P(B_2) \cdots P(B_n)
\]

\[
= P(A_i^c)P(B_2) \cdots P(B_n);
\]

thus we have proved that

\[
P(C_1 \cap B_2 \ldots \cap B_n) = P(C_1)P(B_2) \cdots P(B_n)
\]

(3)

for \( C_1 \in A_i \), and \( B_j \in \{ \emptyset, A_i, \Omega \} \) for \( j = 2, \ldots, n \). This is the first step of an induction argument. Now suppose that for some \( k \in \{ 1, \ldots, n \} \),

\[
P(C_1 \cdots C_{k-1} \cap B_k \cdots B_n) = P(C_1) \cdots P(C_{k-1})P(B_k) \cdots P(B_n)
\]

(4)

for all \( C_i \in A_i \), \( i = 1, \ldots, k-1 \), \( B_i \in \{ \emptyset, A_i, \Omega \} \), \( i = k, \ldots, n \). Since \( \Omega = A_k + A_k^c \) it follows that

\[
\Omega \cap_{j=1}^{k-1} C_j \cap_{j=k+1}^n B_j = A_k \cap_{j=1}^{k-1} C_j \cap_{j=k+1}^n B_j + A_k^c \cap_{j=1}^{k-1} C_j \cap_{j=k+1}^n B_j.
\]

Thus upon replacing \( B_k \) by \( A_k^c \) on the left side of (4), we see that we have, since both \( \Omega, A_k \in \{ \emptyset, A_k, \Omega \} \),

\[
P(\cap_{j=1}^{k-1} C_j \cap A_k^c \cap_{j=k+1}^n B_j)
\]

\[
= P(\cap_{j=1}^{k-1} C_j \cap A_k^c \cap_{j=k+1}^n B_j) - P(\cap_{j=1}^{k-1} C_j \cap A_k \cap_{j=k+1}^n B_j)
\]

\[
= \prod_{j=1}^{k-1} P(C_j)P(\Omega) \prod_{j=k+1}^n P(B_j) - \prod_{j=1}^{k-1} P(C_j)P(A_k) \prod_{j=k+1}^n P(B_j)
\]

\[
= \prod_{j=1}^{k-1} P(C_j) \cdot (1 - P(A_k)) \cdot \prod_{j=k+1}^n P(B_j)
\]

\[
= \prod_{j=1}^{k-1} P(C_j) \cdot P(A_k^c) \cdot \prod_{j=k+1}^n P(B_j).
\]
Hence we have proved that (4) implies that

\[ P(C_1 \cdots C_k B_{k+1} \cdots B_n) = P(C_1) \cdots P(C_k) P(B_{k+1}) \cdots P(B_n) \]  
(5)

for all \( C_i \in \mathcal{A}_i, \ i = 1, \ldots, k, \ B_i \in \{\emptyset, \Omega\}, \ i = k + 1, \ldots, n \). It then follows by induction that

\[ P(C_1 \cdots C_n) = P(C_1) \cdots P(C_n) \]  
(6)

for all \( C_i \in \mathcal{A}_i, \ i = 1, \ldots, n \); i.e. the \( \sigma \)-fields \( \mathcal{A}_i, \ i = 1, \ldots, n \) are independent.

(a) This follows immediately from (a) with \( n = 2 \).

2. Give an example of two collections of sets \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) that are independent but the generated \( \sigma \)-fields are not independent.

**Solution:** One example of this goes as follows: let \( \Omega = \{1, 2, 3, 4\} \), and let \( \mathcal{A} = 2^\Omega \). Suppose that \( P(\{\omega\}) = 1/4 \) for each \( \omega \in \Omega \). Suppose that \( \mathcal{A}_1 = \{\{1, 2\}\} \) and \( \mathcal{A}_2 = \{\{2, 3\}, \{2, 4\}\} \). For \( A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2 \), we have

\[ P(A_1 \cap A_2) = P(\{2\}) = 1/4 = (1/2)(1/2) = P(A_1)P(A_2) \]

so that \( \mathcal{A}_1, \mathcal{A}_2 \) are independent classes. Then we have \( \{2\} \in \sigma[\mathcal{A}_2] \) (since \( \{2\} = \{2, 3\} \cap \{2, 4\} \)) and \( \{1, 2\} \in \sigma[\mathcal{A}_1] \), but

\[ P(\{1, 2\} \cap \{2\}) = P(\{2\}) = 1/4 \neq 1/8 = P(\{1, 2\})P(\{2\}) \cdot \]

Thus \( \sigma[\mathcal{A}_1] \) and \( \sigma[\mathcal{A}_2] \) are not independent classes. The difficulty here is that the class \( \mathcal{A}_2 \) is not a \( \pi \)-system.

3. Show that if \( X_n \) is any sequence of random variables, there are constants \( c_n \to \infty \) so that \( X_n/c_n \to_{a.s.} 0 \).

**Solution:** Define \( F_n(x) \equiv P(|X_n| \leq x) \), the distribution function of \( |X_n| \). Set \( b_n = F_n^{-1}(1 - n^{-2}) \) for \( n = 1, 2, \ldots, \) and let \( \{a_n\} \) be any sequence with \( a_n \to \infty \). Let \( c_n = a_n b_n \). Then, for any \( \epsilon > 0 \) we have \( \epsilon a_n \geq 1 \) for \( n \geq N_\epsilon \) and hence, using \( F_n \circ F_n^{-1}(t) \geq t \) for all \( 0 < t < 1 \),

\[ P(|X_n| > \epsilon c_n) = P(|X_n| > \epsilon a_n b_n) \leq P(|X_n| > b_n) \leq 1 - F_n(b_n) = 1 - F_n(F_n^{-1}(1 - n^{-2})) \leq n^{-2}, \ n \geq N_\epsilon. \]
Hence by the first Borel-Cantelli lemma, $P(|X_n| > \epsilon c_n)$ i.o.) = 0 for every $\epsilon > 0$; that is, $X_n/c_n \to_{a.s.} 0$.

4. Show that if $P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$, then $P(A_n \text{i.o.}) = 0$.

**Solution 1:** Let $B_n \equiv A_n \cap A_{n+1}^c$. Then by the first Borel-Cantelli lemma, $P(\limsup_n B_n) = P(B_n \text{i.o.}) = 0$. Furthermore, since $P(A_n) \to 0$ it follows that $P(\limsup_n A_{n+1}^c) = 0$. Suppose that the following claim holds:

**Claim:** $\limsup_n A_n \cap \limsup_n A_{n+1}^c \subset B_n$.

Then

$$P(\limsup_n A_n) = P(\limsup_n A_n \cap \limsup_n A_{n+1}^c) + P(\limsup_n A_n \cap [\limsup_n A_{n+1}^c]) \leq P(\limsup_n B_n) + P([\limsup_n A_{n+1}^c]) = 0 + 0 = 0.$$ 

To prove the claim, let $\omega \in \limsup_n A_n \cap (\limsup_n A_{n+1}^c)$. Fix $N \geq 1$. Then there exists an $m \geq N$ such that $\omega \in A_m$. Now let $t = \inf\{k > m : \omega \in A_k^c\}$. Such a $t$ exists since $\omega \in \limsup_n A_{n+1}^c$. But then $\omega \in A_t \cap A_{t+1}^c$, and hence

$$\omega \in \cap_{n=1}^{N} \cup_{m=n}^{\infty} (A_m \cap A_{m+1}^c).$$

Since this holds for any $N$ we have $\omega \in \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} (A_m \cap A_{m+1}^c) = \limsup_n B_n$.

**Solution 2:** First we write

$$[A_n \text{i.o.}] = \lim_{n} \cup_{m=n}^{\infty} A_m = \lim_{n} N \cup_{m=n}^{N} A_m$$

and hence

$$P[A_n \text{i.o.}] = \lim_{n} N P(\cup_{m=n}^{N} A_m).$$

Now write

$$\cup_{m=n}^{N} A_m = (\cup_{m=n}^{N-1} (A_m \cap A_{m+1}^c)) \cup A_N$$
and hence
\[ P(\bigcup_{m=n}^{N} A_m) \leq P\left(\bigcup_{m=n}^{N-1} (A_m \cap A_{m+1}^c)\right) + P(A_N). \]

This yields
\[
P[A_n \text{ i.o.}] = \lim_{n} \lim_{N} N P(\bigcup_{m=n}^{N} A_m) \\
\leq \lim_{n} \lim_{N} N \left\{ P\left(\bigcup_{m=n}^{N-1} (A_m \cap A_{m+1}^c)\right) + P(A_N) \right\} \\
\leq \lim_{n} \sum_{m=n}^{\infty} P(A_m \cap A_{m+1}^c) + \lim_{N \to \infty} P(A_N) \\
= 0 + 0 = 0.
\]

5. Let \( X_1, X_2, \ldots \) be independent. Show that \( \sup_{n \geq 1} X_n < \infty \) almost surely if and only if \( \sum_{n=1}^{\infty} P(X_n > M) < \infty \) for some \( M < \infty \).

**Solution:** Suppose that \( \sum_{n} P(X_n > M) < \infty \) for some \( M < \infty \). Then by the first Borel-Cantelli lemma, \( P(X_n > M \text{ i.o.}) = 0 \); i.e. for \( n \geq N_\omega \) we have \( X_n(\omega) \leq M \). Thus
\[
\sup_{n} X_n(\omega) \leq \left( \max_{1 \leq k < N_\omega} X_k \right) \vee M < \infty.
\]

Now suppose that \( \sup_{n} X_n < \infty \) almost surely. If \( \sum_{n} P(X_n > M) = \infty \) for every \( M < \infty \), then, by the second Borel-Cantelli lemma, \( P(X_n > M \text{ i.o.}) = 1 \) for every \( M \); i.e. \( \lim_{n \to \infty} \sup_{n} X_n \geq M \) a.s. for every \( M > 0 \), and this implies, by taking a sequence \( M_k \nearrow \infty \), that \( \lim_{n \to \infty} \sup_{n} X_n = \infty \) a.s., which contradicts \( \sup_{n} X_n < \infty \) almost surely. We therefore conclude that \( \sum_{n} P(X_n > M) < \infty \) for some \( M < \infty \).

6. Suppose that \( X_1, X_2, \ldots \) are independent with \( P(X_n > x) = x^{-r} \) for all \( x \geq 1 \) and \( n = 1, 2, \ldots \) with \( r > 0 \). Show that \( \lim \sup_{n \to \infty} (\log X_n)/\log n = c \) almost surely for some number \( c \), and find \( c \).

**Solution:** Let \( c > 0 \). Then
\[
P(\log X_n > c \log n) = P(X_n > n^c) = n^{-5r}.
\]

Hence by the first and second Borel-Cantelli lemmas
\[
P(\log X_n > c \log n \text{ i.o.}) = \begin{cases} 
0 & \text{if } c > 1/r \\
1 & \text{if } c \leq 1/r
\end{cases}.
\]
Hence
\[
\limsup_{n \to \infty} \left( \frac{\log X_n}{\log n} \right) = \frac{1}{r} \quad \text{almost surely}.
\]