Statistics 522, Problem Set 1 Solutions
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1. PfS, Exercise 8.4.17, page 168:
   (a) Let $X, X_1, \ldots, X_n$ be i.i.d. with the distribution function $F$. Let $F$ have finite mean $\mu = E(X)$. We know $M_n \equiv \max_{1 \leq k \leq n} |X_{n,k}|/n \to_p 0$ by the WLLN. Trivially $E(M_n) \leq E|X|$. Show that $E(M_n) = E(\max_{1 \leq k \leq n} |X_{n,k}|/n) \to 0$.
   (b) Let $X_1, X_2, \ldots$ be i.i.d. Show that $E|X| < \infty$ if and only if $M_\infty = \max_{1 \leq k \leq n} |X_{n,k}|/n \to 0$.

Solution: (a) Note that $M_n \leq n^{-1} \sum_{k=1}^n |X_{n,k}| \equiv Y_n$ where $Y_n \to_p E|X|$ and where $E(Y_n) = E|X|$ satisfies $\limsup_n E(Y_n) = E|X|$. Thus by Vitali’s theorem $\{Y_n\}$ is uniformly integrable. Since $M_n \leq Y_n$ we conclude that $\{M_n\}$ is uniformly integrable: $E(M_n 1_{[M_n \geq \lambda]}) \leq E(Y_n 1_{[Y_n \geq \lambda]})$ and hence

$$\limsup_{n \to \infty} \frac{E(M_n 1_{[M_n \geq \lambda]})}{n} \leq \limsup_{n \to \infty} \frac{E(Y_n 1_{[Y_n \geq \lambda]})}{n} \to 0$$

as $\lambda \to \infty$. Since $M_n \to_p 0$ and is uniformly integrable, Vitali’s theorem yields $E(M_n) \to 0$; i.e. $M_n \to 0$.
   (b) Now $M_n = n^{-1} \max_{1 \leq k \leq n} |X_k|$ where the $X_k$’s are i.i.d. If $E|X_1| < \infty$, then, as in (a) $M_n \leq n^{-1} \sum_{k=1}^n |X_k| \equiv Y_n$ where $Y_n \to a.s. E|X|$ and where $E(Y_n) = E|X|$ satisfies $\limsup_n E(Y_n) = E|X|$. Thus by Vitali’s theorem $\{Y_n\}$ is uniformly integrable. Since $M_n \leq Y_n$ we conclude that $\{M_n\}$ is uniformly integrable: $E(M_n 1_{[M_n \geq \lambda]}) \leq E(Y_n 1_{[Y_n \geq \lambda]})$ and hence

$$\limsup_{n \to \infty} \frac{E(M_n 1_{[M_n \geq \lambda]})}{n} \leq \limsup_{n \to \infty} \frac{E(Y_n 1_{[Y_n \geq \lambda]})}{n} \to 0$$

as $\lambda \to \infty$. Since $M_n \to a.s. 0$ and is uniformly integrable, Vitali’s theorem yields $E(M_n) \to 0$; i.e. $M_n \to 0$.

Now suppose that $M_n \to 0$. Suppose that $E|X| = \infty$. Then

$$E \max_{1 \leq i \leq n} \frac{|X_i|}{n} \geq E \left\{ \frac{|X_n|}{n} \right\} = E \left\{ \frac{|X_1|}{n} \right\} = \infty.$$

But since the left side is converging to 0, this is a contradiction. Hence $E|X_1| < \infty$. 

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2. PfS, Exercise 8.5.2, page 174: (Monte Carlo estimation)

Let \( h : [0, 1] \to [0, 1] \) be continuous.

(i) Let \( X_k \equiv 1\{h(\xi_k) \geq \Theta_k\} \) where \( \xi_1, \xi_2, \ldots, \Theta_1, \Theta_2, \ldots \) are i.i.d. Uniform(0, 1) rv's. Show that the sample average \( X_n \) is a strongly consistent estimator of the integral \( \int_0^1 h(t)dt \); i.e. \( X_n \to a.s. \int_0^1 h(t)dt \).

(ii) Let \( Y_k \equiv h(\xi_k) \). Show that \( Y_n \to a.s. \int_0^1 h(t)dt \).

(iii) Evaluate \( \text{Var}(X_n) \) and \( \text{Var}(Y_n) \) and compare them.

Solution: (i) Now

\[ EX_1 = E1_{\{h(\xi_1) \geq \Theta_1\}} = \int_0^1 \left( \int_0^1 1_{\{h(v) \geq u\}}du \right) dv = \int_0^1 h(v)dv < \infty \]

since \( 0 \leq h \leq 1 \). Thus the SLLN yields \( X_n \to a.s. EX_1 = \int_0^1 h(v)dv \).

(ii) Here \( EY_1 = Eh(\xi_1) = \int_0^1 h(v)dv \) and the SLLN again yields \( Y_n \to a.s. EY_1 \).

(iii) \( \text{Var}(X_1) = E(X_1^2) - (E(X_1))^2 = \int_0^1 \int_0^1 1_{\{h(v) \geq u\}}dudv - \left( \int_0^1 h(t)dt \right)^2 \]

\[ = \int_0^1 h(t)dt - \left( \int_0^1 h(t)dt \right)^2 \equiv \bar{h}(1 - \bar{h}) \]

where \( \bar{h} \equiv \int_0^1 h(t)dt \in [0, 1] \). Furthermore,

\[ \text{Var}(Y_1) = E(Y_1^2) - (E(Y_1))^2 = \int_0^1 h^2(t)dt - \left( \int_0^1 h(t)dt \right)^2 \]

where \( \int_0^1 h^2(t)dt \leq \int_0^1 h(t)dt \) since \( 0 \leq h(t) \leq 1 \) for all \( t \in [0, 1] \). Thus \( \text{Var}(Y_1) \leq \text{Var}(X_1) \) and \( Y_n \) yields an estimator of \( \int_0^1 h(t)dt \) with smaller variance than \( X_n \).

3. PfS, Exercise 8.8.1, page 182: Suppose that \( X_1, X_2, \ldots \) are i.i.d. with \( P(X_k = 0) = P(X_k = 2) = 1/2 \). Show that \( S_n \equiv \sum_{k=1}^n X_k/3^k \to a.s. \) some \( S \), and determine the mean, variance, and the name of the d.f. \( F_S \) of \( S \).
Solution: Note that

\[ E(S_n) = \sum_{k=1}^{n} \frac{E(X_k)}{3^k} = \sum_{k=1}^{n} \frac{1}{3^k} \rightarrow \sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1}{1 - 1/3} = \frac{1}{2}, \]

\[ Var(S_n) = \sum_{k=1}^{n} \frac{Var(X_k)}{9^k} = \sum_{k=1}^{n} \frac{1}{9^k} \rightarrow \sum_{k=1}^{\infty} \frac{1}{9^k} = \frac{1/9}{1 - 1/9} = \frac{1}{8}. \]

Thus by the two-series theorem it follows that \( S_n \rightarrow S \equiv \sum_{k=1}^{\infty} X_k / 3^k \).

This has distribution function \( F_S \) given by the Cantor singular distribution function on \([0, 1]\); recall Example 6.1.1, page 105.

4. PfS, Exercise 8.8.3, page 182: Suppose that \( X_1, X_2, \ldots \) are arbitrary random variables with all \( X_k \geq 0 \) a.s. Let \( c > 0 \) be arbitrary. Then \( \sum_{k=1}^{\infty} E(X_k \wedge c) < \infty \) implies that \( \sum_{k=1}^{n} X_k \rightarrow a.s. \) (some rv \( S \)). The converse holds for independent random variables.

Solution: (a) Note that:

1. Since all \( X_k \geq 0 \), \( \max_{n \leq m \leq N} \sum_{k=n}^{m} X_k \leq \sum_{k=n}^{N} X_k \).
2. For any \( c > 0 \), \( [X_k \geq c] \subset [X \wedge c \geq c] \).

From (2) it follows that

\[ P(X_k \geq c) \leq P(X_k \wedge c \geq c) \leq \frac{E(X_k \wedge c)}{c}, \]

and hence \( \sum_{k=1}^{\infty} P(X_k \geq c) \leq c^{-1} \sum_{k=1}^{\infty} E(X_k \wedge c) < \infty \). Thus \( P(X_k \geq c \ i.o.) = 0 \) by the first Borel-Cantelli lemma. Thus \( \{X_k\} \) and \( \{X_k \mid X_k \leq c\} \equiv X_k^{(c)} \) are Khinchine equivalent, and it suffices to show that \( \sum_{k=1}^{n} X_k^{(c)} \rightarrow a.s. \) some \( S^{(c)} \). But by (1) with \( X_k \) replaced by \( X_k^{(c)} \),

\[ P(\max_{n \leq m \leq N} \sum_{k=n}^{m} X_k^{(c)} > \epsilon) = P(\sum_{k=n}^{N} X_k^{(c)} > \epsilon) \]

\[ \leq \epsilon^{-1} E(\sum_{k=n}^{N} X_k^{(c)}) \leq \epsilon^{-1} \sum_{k=n}^{N} E(X_k \wedge c) \]

\[ \leq \epsilon^{-1} \sum_{k=n}^{\infty} E(X_k \wedge c) \rightarrow 0 \]

as \( n \rightarrow \infty \).

(b) Suppose that \( S_n \equiv \sum_{k=1}^{n} X_k \rightarrow a.s. \) \( S \) where \( X_1, X_2, \ldots \) are independent. By the three-series theorem (Theorem 8.3, PfS, page 181), the
three series $I_c \equiv \sum_{k=1}^{\infty} P(|X_k| > c)$, $II_c \equiv \sum_{k=1}^{\infty} Var(X_k^{(c)})$, and $III_c \equiv \sum_{k=1}^{\infty} E X_k^{(c)}$ all converge for every $c > 0$ where $X_k^{(c)} \equiv X_k I_{|X_k| \leq c}$. But since $X_k \geq 0$,

$$X_k \wedge c = \begin{cases} X_k, & X_k \leq c \\ c, & X_k > c \end{cases} = X_k^{(c)} + c1_{[X_k>c]}.$$

Taking expectations across this identity and summing on $k$ yields

$$\sum_{k=1}^{\infty} E(X_k \wedge c) = \sum_{k=1}^{\infty} \left\{ EX_k^{(c)} + cP(X_k > c) \right\} = III_c + cII_c < \infty$$

for every $c > 0$. 