Statistics 523, Problem Set 2
Wellner; 4/5/17

Reading: Shorack, PfS Course Notes, Chapter 10, pages 225-253;
Shorack, PfS Course Notes, Chapter 11, pages 273-287.

Due: Wednesday, April 12, 2017.

1. PfS Course Notes, Exercise 10.2.1, page 236. (Characterization of “uan”) Suppose that $\{X_{n,k} : 1 \leq k \leq n\}$ is a row-independent triangular array with $E(X_{n,k}) = 0$, $E(X_{n,k}^2) = \sigma_{n,k}^2$, normalized so that $\sigma_n^2 \equiv \sum_{k=1}^{n} \sigma_{n,k}^2 = 1$. Show that the following are equivalent:
   (a) $|X_{n,k}|$'s are uan; that is, $\max_{1 \leq k \leq n} P(|X_{n,k}| \geq \epsilon) \to 0$ for all $\epsilon > 0$.
   (b) $\max_{1 \leq k \leq n} |\phi_{nk}(t) - 1| \to 0$ uniformly on every finite interval of $t$'s.
   (c) $\max_{1 \leq k \leq n} E(X_{n,k}^2 \wedge 1) = \max_{1 \leq k \leq n} \int (x^2 \wedge 1) dF_{nk}(x) \to 0$.

2. PfS Course Notes, Exercise 10.2.8, page 237.
   (i) Show that Lindeberg’s condition that $LF_n(\epsilon) \to 0$ for all $\epsilon > 0$ implies Feller’s condition that $\max_{1 \leq k \leq n} \sigma_{n,k}^2 / \sigma_n^2 \to 0$.
   (ii) Let $X_{n1}, \ldots, X_{nn}$ be row independent Poisson($\lambda/n$) random variables with $\lambda > 0$. Discuss which of the Lindeberg-Feller, Liapunov, and Feller conditions holds in this context. [The Liapunov ($2 + \delta$) condition is as follows: for some $0 < \delta \leq 1$ we have]
   \[ \sum_{k=1}^{n} E|X_{nk} - \mu_{nk}|^{2+\delta} / \sigma_n^{2+\delta} \to 0. \]
   (iii) Repeat part (ii) when $X_{n1}, \ldots, X_{nn}$ are row independent and all have the probability density $cx^{-3}(\log x)^2$ on $x \geq 3$ (for some constant $c > 0$).
   (iv) Repeat part (ii) when $P(X_{nk} = a_k) = P(X_{nk} = -a_k) = 1/2$ for row-independent $X_{nk}$’s. Discuss this for general $a_k$’s and present two or three interesting examples for which the various conditions differ (i.e. hold or fail to hold).

3. Suppose that $\{X_k : 1 \leq k < \infty\}$ are independent random variables with $P(X_k = \pm k) = 1/(2k^2)$ and (for $k \geq 2$) $P(X_k = \pm 1) = (1 - (k^{-2})/2$. Let $S_n = \sum_{k=1}^{n} X_k$.
   (a) Show that $\text{Var}(S_n)/n \to 2$.
   (b) Compute $\max_{1 \leq k \leq n} \text{Var}(X_k)/\text{Var}(S_n)$ and show that it converges to 0.
   (c) Does the Lindeberg-Feller condition hold?
   (d) Does $S_n/\sqrt{\text{Var}(S_n)} \to N(0,1)$?
4. Suppose that \( \{X_k : k \geq 1\} \) are independent random variables with
\[
P(X_k = \pm k^\alpha) = \frac{1}{6k^{2(\alpha-1)}} \quad \text{and} \quad P(X_0 = 0) = 1 - \frac{1}{3k^{2(1-\alpha)}}.
\]
Show that the Lindeberg condition holds if and only if \( \alpha < 3/2 \).

5. \( \{X_k : k \geq 1\} \) satisfies a Lindeberg condition of order \( r \) if
\[
\frac{1}{s_n^r} \sum_{k=1}^n E\{|X_k|^r 1_{|X_k| > \epsilon s_n}\} \to 0
\]
for every \( \epsilon > 0 \) where \( s_n^2 \equiv \sum_{k=1}^n \sigma_k^2 \). Suppose that \( \{X_k : k \geq 1\} \) are independent random variables with \( E(X_k) = 0 \), \( E(X_k^2) = \sigma_k^2 < \infty \). Show that if \( \{X_k\} \) satisfies a Lindeberg condition of order \( r \) for some integer \( r \geq 2 \), then \( E(S_n/s_n)^k \to EZ^k \) for each \( k = 1, 2, \ldots, r \) where \( Z \sim N(0,1) \).

6. Optional bonus problem 1: Suppose that \( \{X_k : k \geq 1\} \) are independent random variables with \( E(X_k) = 0 \), \( E(X_k^2) = \sigma_k^2 < \infty \). Suppose that \( \{X_k : k \geq 1\} \) are independent random variables with \( E(X_k) = 0 \), \( E(X_k^2) = \sigma_k^2 < \infty \), and \( S_n/s_n \to_d Z \sim N(0,1) \), and \( E\{(s_n^{-1}S_n)^{2m}\} = (2m)!/(2^m m!) \) with \( s_n^2 \equiv \sum_{k=1}^n \sigma_k^2 \).
(a) Show that \( \{X_k : k \geq 1\} \) satisfies a Lindeberg condition of order \( 2m \).
(b) Suppose that \( \{X_k : k \geq 1\} \) satisfies a Lindeberg condition of order \( r > 2 \). Show that this implies that \( \sum_{k=1}^E |X_k|^r = o(s_n^r) \).

7. Optional bonus problem 2: Suppose that \( T \sim \text{Poisson}(\lambda) \). (a) Show that \( (T - \lambda)/\sqrt{\lambda} \to_d Z \sim N(0,1) \) as \( \lambda \to \infty \).
(b) Suppose that \( \{X_{n,k} : k \geq 1\} \) is a triangular array of independent Poisson random variables with parameters \( \{\lambda_{n,k} : k \geq 1\} \). Suppose that \( \lambda_n \equiv \sum_{k=1}^n \lambda_{n,k} \) where \( \lambda_n \to \infty \). In view of (a) it is natural to conjecture that \( T_n \equiv \sum_{k=1}^n X_{n,k} \) satisfies \( (T_n - \lambda_n)/\sqrt{\lambda_n} \to_d Z \sim N(0,1) \). Are any other conditions needed on the \( \lambda_{n,k} \)'s to prove this?