Reading: Shorack, PfS Course Notes, Chapter 12, pages 312-332; Durrett, PTE, Chapter 8, pages 353 - 396.

Reminder: Make-up lecture 2: Wednesday, April 26, 12:30 - 1:20, CMU 230.
Reminder: Tentative paper/project outlines due next Wednesday, May 3.
Reminder: Mid-term exam: Friday, May 12.

Due: Wednesday, May 3, 2017.

1. Suppose that $a_n \nearrow$ with $a_1 = 1$ and suppose that $a_{mk} = a_m a_k$ for all $k, m \geq 1$. Show that $a_n = n^{1/\alpha}$ for some $\alpha \geq 0$.

2. Assume that $Y$ with distribution function $G$ is stable with characteristic exponent $\alpha$. Show that $E|Y|^r < \infty$ for all $0 < r < \alpha$. [Hint: use the inequalities of PfS Section 8.3 to show that $nP(|X| > a_n x)$ is bounded in $n$, where $a_n \equiv n^{1/\alpha}$. Then bound the appropriate integral.

3. Assume that $n$ objects $X_{n,1}, \ldots, X_{n,n}$ are placed independently and at random in $[-n,n]$. Let

$$Z_n = \sum_{k=1}^{n} \frac{\text{sign}(X_{n,k})}{|X_{n,k}|^p}$$

be the total force exerted on 0.

(i) Show that if $p > 1/2$, then $E \exp(itZ_n) \to \exp(-c|t|^{1/p})$ for $t \in \mathbb{R}$ for some $c$.

(ii) Show that if $p < 1/2$, then $Z_n/n^{1/2-p} \to_d cZ$ where $Z \sim N(0,1)$.

(iii) Show that if $p = 1/2$, then $Z_n/(\log n)^{1/2} \to cZ$.

Hint: see Durrett, PTE, Example 3.7, page 165, and Exercise 3.7.3 page 166.

4. Let $X_1, X_2, \ldots$ be i.i.d. with a density function $f$ that is symmetric about 0 and continuous and positive at 0. Show that

$$\frac{1}{n} \left( \frac{1}{X_1} + \cdots + \frac{1}{X_n} \right) \to_d Y$$

5. (i) Suppose that \( X \) is symmetric stable with index \( \alpha \) and \( Y \geq 0 \) is an independent stable with index \( \beta < 1 \), then \( XY^{1/\alpha} \) is symmetric stable with index \( \alpha\beta \).

(ii) Suppose that \( Z_1 \) and \( Z_2 \) are independent \( N(0, 1) \) random variables. Check that \( 1/Z_1^2 \) has the density of \( T_1 \) the first time Brownian motion starting from 0 hits the level 1. Use the to show that \( Z_1/Z_2 \) has a Cauchy distribution.

6. **Bonus problem 1:** Suppose that \( \{X(t) : t \geq 0\} \) is a process with stationary and independent increments with \( X(0) = 0 \) and characteristic function of \( X(t) \) given by

\[
Ee^{iuX(t)} = \exp(-tc|u|^\alpha \{1 - i\text{sign}(u)C_\alpha\})
\]

where \( \alpha \in (0, 1) \), \( c \geq 0 \) and \( C_\alpha = \tan(\pi\alpha/2) \). Thus the (marginal, or one-dimensional) distributions of \( X(t) \) are completely asymmetric stable laws with exponent \( \alpha \in (0, 1) \).

(a) Show that \( X(t) \overset{d}{=} t^{1/\alpha}X(1) \) for all \( t > 0 \).

(b) Let \( 0 < r < \alpha \). Use (a) to compute \( E|X(t)|^r \) in terms of \( E|X(1)|^r \) where the latter is finite by Problem 2.

7. **Bonus problem 2:** Now let \( S \) be a standard Brownian motion on \([0, \infty)\), let \( X(t) \) be a completely asymmetric stable process (sometimes called a **stable subordinator**) of index \( \alpha \in (0, 1) \) as in bonus problem 1 above which is independent of \( S \). Consider the new process \( Y(t) \equiv S(X(t)) \) for \( t \geq 0 \).

(a) Use a calculation similar to that of Problem 5 to show that \( Y \) is a symmetric stable process of index \( 2\alpha \).

(b) Does \( Y \) have stationary independent increments?

8. **Bonus problem 3:** Drop the symmetry assumption in Problem 4, but assume that the density \( f \) is differentiable at 0. Does the conclusion of Problem 4 still hold?