1. (a) Give an example of a random variable $Y$ with distribution function $F$ on $\mathbb{R}^+ = [0, \infty)$ for which $EY^r = \infty$ for all $r > 0$.
(b) Does your example in (a) have $Eg(Y) < \infty$ for some measurable function $g$ with $g(y) \to \infty$ as $y \to \infty$?

Solution: (a) Suppose that $F$ is defined by
$$
1 - F(x) = \begin{cases} 
\frac{(\log x)^{-\gamma}}{\gamma}, & x \geq e, \\
1, & x < e,
\end{cases}
$$
where $\gamma > 0$. Note that $F$ has density $f$ given by $f(x) = \gamma x^{-1}(\log x)^{-1-\gamma}1_{[e,\infty)}(x)$.
Then if $Y \sim F$,
$$
EY^r = \int_0^\infty rx^{r-1}(1-F(x))dx = \int_0^e rx^{r-1}dx + \int_e^\infty rx^{r-1}(\log x)^{-\gamma}dx
$$
$$
= \infty \quad \text{for all} \quad r > 0
$$
since $\lim_{x \to \infty} x^r/(\log x)^\gamma = \infty$ for all $r, \gamma > 0$.
(b) Consider $g(x) = (\log x)^\delta$. Then $g(x) \to \infty$ if $\delta > 0$. Moreover,
$$
Eg(Y) = \int_e^\infty (\log x)^\delta \frac{\gamma}{x(\log x)^{\gamma+1}}dx
$$
$$
= \gamma \int_1^\infty v^{-(1+\gamma-\delta)}dv = \frac{\gamma}{\gamma - \delta} < \infty
$$
if $0 < \delta < \gamma$.

2. A very famous theorem conjectured by Lévy and proved by Cramér (1936) says that if $X$ and $Y$ are independent random variables with $X + Y = Z$ having a Normal distribution, then both $X$ and $Y$ have normal distributions. Find a statement and proof of this theorem. What are the crucial ingredients of the proof?


\[1\]

The proof depends on the following interesting lemma concerning characteristic functions:

**Lemma:** Let $F$ be a probability distribution such that

$$g(\eta) = \int_{-\infty}^{\infty} \exp(\eta^2 x^2) dF(x) < \infty$$

for some $\eta > 0$. Then the characteristic function $\varphi$ of $F$ is an entire function defined for all complex $z$. If $\varphi(z) \neq 0$ for all complex $z$, then $F$ is normal.

An equivalent modern statement of the exponential integrability hypothesis would be to say that $X \sim F$ has a finite $\psi_2$–Orlicz norm $\|X\|_{\psi_2}$ where $\psi_2(x) = \exp(x^2) - 1$.

Another statement of the Theorem (from Chow and Teicher (1978), page 282) is as follows:

**Theorem:** (Cramér - Lévy) The family of normal distributions is factor closed. It turns out that the families of Poisson distributions and binomial distributions are also factor closed (to within translations); see Chow and Teicher (1978), Theorem 4, page 283.

3. Stein’s method for convergence in distribution to the Poisson distribution depends on the following characterization: $X \sim \text{Poisson}(\lambda)$ if and only if

$$E[X f(X)] = \lambda E[f(X + 1)]$$

for all functions $f$ for which the expectations exist. Show that if $X \sim \text{Poisson}(\lambda)$ then the identity in the display holds for any bounded function $f$.

**Proof.** Suppose that $X \sim \text{Poisson}(\lambda)$ and let $f : \mathbb{N} \to \mathbb{R}$ be bounded. Then

$$E[X f(X)] = \sum_{k=0}^{\infty} k f(k) e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} f(k) e^{-\lambda} \frac{\lambda^k}{(k-1)!}$$

$$= \sum_{m=0}^{\infty} f(m+1) e^{-\lambda} \frac{\lambda^{m+1}}{m!} = \lambda \sum_{m=0}^{\infty} e^{-\lambda} \frac{\lambda^m}{m!}$$

$$= \lambda E[f(X + 1)].$$

The reverse argument goes as follows: Suppose that for any $A \subset \mathbb{N}$ we can construct a function $g_{A,\lambda} : \mathbb{N} \to \mathbb{R}$ satisfying

$$\lambda g(k+1) - kg(k) = 1_A(k) - \text{Poiss}_\lambda(A)$$  \hspace{1cm} (1)
for all $k \geq 0$. Then if $W$ takes values in $\mathbb{N}$ we have

$$\lambda E\{\lambda g(W + 1) - W g(W)\} = P(W \in A) - \text{Poiss}_\lambda(A)$$

If the left side is zero, then we conclude that $W$ has a Poisson$(\lambda)$ distribution. The solution of (1) can be found recursively, starting from $k = 0$ and working up.

4. Goldstein’s probabilistic proof of the Lindeberg-Feller CLT relies on the following lemma, which is a kind of converse for Slutsky’s lemma.

Let $\{U_n\}$ and $\{V_n\}$ be sequences of random variables such that $U_n$ and $V_n$ are independent for every $n$. Then $U_n \to U$ and $U_n + V_n \to_d U$ implies that $V_n \to_p 0$. Prove this lemma. (This is Lemma 5.1 in Goldstein (2009).)

Solution: As I noted in class on 17 April, Independence of $U_n$ and $V_n$ for each $n$ together with convergence in distribution of both $U_n$ and $U_n + V_n$ to $U$ yields

$$\phi_U(t) = E e^{itU} \leftarrow E e^{it(U_n + V_n)} = E e^{itU_n} \cdot E e^{itV_n} \to \phi_U(t) \cdot \lim_{n \to \infty} E e^{itV_n},$$

and hence $\phi_U(t) \lim_{n \to \infty} E e^{itV_n} = \phi_U(t)$ for all $t \in \mathbb{R}$.

Two of you turned this into a proof by arguing as follows: Since $\phi_U$ is a characteristic function (of a proper random variable), there is a neighborhood of 0, say $|t| < \delta$, such that $\phi_U(t) \neq 0$ for all $|t| < \delta$; this follows from $\phi_U(0) = 1$ and the continuity of $\phi_U$. This leads to the conclusion that the limit $\phi_V(t) = \lim_{n \to \infty} E e^{itV_n} = 1$ for $|t| < \delta$ for some (perhaps small) $\delta > 0$. But this implies that $E(V) = 0$ and $E|V|^2 = 0$ by Durrett (2010), exercise 3.3.19. This implies $V = 0$ with probability 1, and hence $\phi_V(t) = 1$ for all $t \in \mathbb{R}$. Thus $V_n \to_d 0$ and this implies that $V_n \to_p 0$.

Alternatively, Use Exercise 3.3.20, Durrett (2010): $V_n \to_d 0$ if and only if $\phi_{V_n}(t) \to 1$ for $|t| < \delta$ for some $\delta > 0$.

Of course the point of Goldstein’s proof is to (completely!) avoid the use of characteristic functions.

Here is the statement and solution of Durrett’s exercise. If $\lim_{t \to 0} t^{-2}(\phi(t) - 1) = c > -\infty$, then $E(X) = 0$ and $E|X|^2 = -2c < \infty$. In particular, if $\phi(t) = 1 + o(t^2)$, then $\phi(t) \equiv 1$.

Solution: $E|X|^2 < \infty$ follows from Theorem 3.3.9, Durrett (2010). By comparison with $\phi(t) = 1 + it\mu - (1/2)t^2\sigma^2 + o(t^2)$ (Theorem 3.3.8, Durrett (2010)), it follows that $\mu = 0$ and $\sigma^2 = -2c$. If $\phi(t) = 1 + o(t^2)$, then $c = 0$ and $X \equiv 0$. 

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