1. A random variable $X$ takes on the values 1, 2, 3, 4 with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_0(x)$</td>
<td>.54</td>
<td>.08</td>
<td>.12</td>
<td>.26</td>
</tr>
<tr>
<td>$p_1(x)$</td>
<td>.22</td>
<td>.16</td>
<td>.36</td>
<td>.26</td>
</tr>
</tbody>
</table>

(a) Find a most powerful test of size $\alpha = .15$ for testing $p_0$ versus $p_1$ and determine its power.
(b) Find a test $\phi$ which minimizes the Bayes risk with $0 - 1$ loss and prior distribution $(\lambda, 1 - \lambda) = (2/5, 3/5)$; i.e. a test $\phi$ which minimizes $\lambda a + (1 - \lambda)b$ where $a = E_0\phi$ and $b = E_1(1 - \phi)$.

2. Continuation of problem 1. For $P_0$ and $P_1$ as given in problem 1, compute $d_{TV}(P_0, P_1)$, $H(P_0, P_1)$, and the affinity $\rho(P_0, P_1) = \int \sqrt{p_0(x)p_1(x)} d\mu$. For the product laws $P_{0n}$ and $P_{1n}$ (corresponding to observation of $X_1, \ldots, X_n$ i.i.d. $P_0$ or $P_1$ respectively) compute $\rho(P_{0n}, P_{1n})$ and $H(P_{0n}, P_{1n})$ for $n = 20, 50, 100$. What does this imply about the test, $\phi_n$ say, based on $X_1, \ldots, X_n$ which minimizes the sum of risks?

3. Optional bonus problem 1: For observations $X = (X_1, \ldots, X_n)$, let $X_{(1)} \leq \ldots \leq X_{(n)}$ denote the order statistics of the $X_i$'s ($X_{(i)} \equiv F_n^{-1}(i/n)$, $i = 1, \ldots, n$) and let $R = (R_1, \ldots, R_n)$ denote the ranks; defined by $X_i = X_{(R_i)}$, $i = 1, \ldots, n$ (if $X_i = X_j$ for some $i < j$, define the ranks by $R_i < R_j$ and $X_i = X_{(R_i)}$).

(a) Suppose that $X_1, \ldots, X_n$ are i.i.d. $F \in \mathcal{F}_{ac}$ (the absolutely continuous df’s $F$ on $R$) with density $f$. Show that the order statistics $X_{(i)} \equiv (X_{(1)}, \ldots, X_{(n)})$ are independent of the ranks $R$ and that the order statistics have joint density $\bar{p}$ given by

$$\bar{p}(x_{(i)}) = n! \prod_{i=1}^{n} f(x_{(i)}), \quad -\infty < x_{(1)} < \ldots < x_{(n)} < \infty$$
while
\[ P(R = r) = \frac{1}{n!}, \quad r \in \Pi \equiv \{ \text{all permutations of } \{1, \ldots, n\}\} . \]

(b) Show that if the density \( f \) of the \( X_i \)'s is log-concave, then the joint density \( \bar{p} \) of the order statistics \( X_{(\cdot)} \) is log-concave; i.e. show that if \( f((x + y)/2)^2 \geq f(x)f(y) \) for all \( x, y \in \mathbb{R} \), then \( \bar{p}((x + y)/2)^2 \geq \bar{p}(x)\bar{p}(y) \) for all \( x, y \in \mathcal{O}_n \equiv \{ x \in \mathbb{R}^n : x_1 \leq x_2 \leq \cdots \leq x_n \} \).

(c) Show that (a) continues to hold for any joint density \( p \) of the \( X \) which is symmetric with respect to permutation of its coordinates: \( p(\pi x) = p(x) \) for all \( x \) and \( \pi \in \Pi \) where \( \pi x \equiv (x_{\pi(1)}, \ldots, x_{\pi(n)}) \).

(d) If the joint density \( p \) of \( X \) is general (not permutation symmetric), show that the joint density \( \bar{p} \) of the order statistics is given by

\[ \bar{p}(x_{(\cdot)}) = \sum_{\pi \in \Pi} p(\pi x_{(\cdot)}) , \]

and

\[ P(R = r | X_{(\cdot)} = x_{(\cdot)}) = \frac{p(r x_{(\cdot)})}{\bar{p}(x_{(\cdot)})} . \]

Hint: The easiest way might be to solve (d) first, then (c) followed by (a) and (b).

4. **Optional bonus problem 2:** Let \( \mathcal{P} = \{ p_\theta : \theta \in \Theta \} \) where \( p_\theta \) is a family of densities with respect to a fixed dominating measure \( \mu \) defined on a sample space \( \mathcal{X} \) and \( \Theta \subset \mathbb{R} \).

(a) Suppose that the densities \( p_\theta(x) \equiv p(x, \theta) \) have a second mixed partial derivative and that

\[ \frac{\partial^2}{\partial x \partial \theta} \log p(x, \theta) \geq 0 \]

for all \( x \in \mathbb{R} \) and \( \theta \in \Theta \). Show that the inequality in the last display implies that \( \mathcal{P} \) has monotone likelihood ratio. [Hint: use the fundamental theorem of calculus twice.]

(b) Show that the condition in (a) is equivalent to

\[ p(x, \theta) \frac{\partial^2}{\partial x \partial \theta} p(x, \theta) \geq \frac{\partial}{\partial \theta} p(x, \theta) \frac{\partial}{\partial x} p(x, \theta) \]

for all \( \theta \in \Theta, x \in \mathcal{X} \).