1. Let $T(F) \equiv \Lambda_F(x_0) = \int_0^{x_0} (1 - F)_-^{-1}dF$ where $1 - F_{x_0}(x) = 1 - F(x -) = \lim_{y \searrow x}(1 - F(y)) = P_F(X \geq x)$ and where $x_0$ is fixed. This is the cumulative hazard function of $F$ at $x_0$.

(a) Is $T$ a weakly continuous function of $F$ (at continuity points of $F$)? Is it continuous with respect to the Kolmogorov (i.e. the uniform metric) on distribution functions?

(b) Find the influence function of $T(F)$. Hint: see van der Vaart, Lemma 20.10, page 298, and Lemma 20.14, page 300.

Solution: (a) No, $T$ is not weakly continuous (at all continuity points of $F$). Suppose that $F_p(x) = (1 - p)1_{[0,\infty)}(x) + p1_{[1,\infty)}(x)$ where $0 \leq p \leq 1$; this is the family of Bernoulli($p$) distributions. Then we compute

$$
\Lambda_p(x) = \int_0^x \frac{1}{1 - F_p^-(x)}dF_p = \begin{cases} 
0, & x < 0 \\
1 - p, & 0 \leq x < 1 \\
(1 - p) + \frac{p}{p} = 2 - p, & 1 \leq x < \infty
\end{cases}
$$

Then with $F_n(x) \equiv F_{p_n}(x)$ where $p_n \to 0$ we have $F_n \to F_0 \equiv 1_{[0,\infty)}$, the distribution which has mass 1 at 0. But for any $x_0 \geq 1$, $x_0$ is a continuity point of $F_0$ and

$$T(F_n) = \Lambda_n(x_0) \equiv \Lambda_{p_n}(x_0) = \begin{cases} 
0, & x_0 < 0 \\
1 - p_n, & 0 \leq x_0 < 1 \\
2 - p_n, & 1 \leq x_0 < \infty
\end{cases}
$$

I have not yet found a counterexample if $x_0 \in \text{int(} \text{supp}(F_0)\text{)}$.

(a), part 2 (Kolmogorov metric): Suppose that $\|F_n - F\|_\infty \to 0$ and assume that $1 - F(x_0-) > 0$ where $x_0$ is a continuity point of $F$. Then, via integration by
parts for the second term,

\[
|T(F_n) - T(F)| = \left| \int_0^{x_0} \frac{dF_n(y)}{1 - F_n(y-)} - \int_0^{x_0} \frac{dF(y)}{1 - F(y-)} \right|
\]

\[
\leq \left| \int_0^{x_0} \left( \frac{1}{1 - F_n(y-)} - \frac{1}{1 - F(y-)} \right) dF_n(y) \right| \\
+ \left| \int_0^{x_0} \frac{1}{1 - F(y-)} d(F_n - F)(y) \right|
\]

\[
= \int_0^{x_0} \frac{|F_n(y) - F(y)|}{(1 - F(y-))(1 - F_n(y-))} dF_n(y)
\]

\[
\leq \int_0^{x_0} \frac{|F_n(y) - F(y)|}{(1 - F(y-))(1 - F_n(y-))} dF_n(y)
\]

The distribution function corresponding to \(G = \delta_x\) is \(1_{(-\infty,y]}(x), y \in \mathbb{R}\), so the left limit is \(G_-(y) = 1_{[x < y]}\), and the corresponding “at risk” function \(1 - G_-(y) = 1_{[x \geq y]}\). Therefore, we have:

\[
\int_0^{x_0} \frac{1}{1 - F(y-)(1 - F_n(y-))} dF_n(y) \leq \frac{1}{(1 - F(x_0-))(1 - F_n(x_0-))} F_n(x_0)
\]

\[
\rightarrow \frac{1}{(1 - F(x_0-))^2} < \infty,
\]

so we have \(T(F_n) \rightarrow T(F)\) if \(x_0\) is a continuity point of \(F\). If \(x_0\) is not a continuity point of \(F\) the same proof apparently works if \(1 - F(x_0) > 0\).

(b) **Solution:** To find the influence function of \(T(F)\), let \(F_t = (1 - t)F + t\delta_x\). The distribution function corresponding to \(G = \delta_x\) is \(1_{(-\infty,y]}(x), y \in \mathbb{R}\), so the left limit is \(G_-(y) = 1_{[x < y]}\), and the corresponding “at risk” function \(1 - G_-(y) = 1_{[x \geq y]}\). Therefore, we have:

\[
\int_0^{x_0} \frac{1}{1 - F(y-)(1 - F_n(y-))} dF_n(y) \leq \frac{1}{(1 - F(x_0-))(1 - F_n(x_0-))} F_n(x_0)
\]

\[
\rightarrow \frac{1}{(1 - F(x_0-))^2} < \infty,
\]
\(1_{[x \geq y]} = 1_{[y, \infty)}(x)\). We need to compute

\[
\lim_{t \to 0} \frac{T(F_t) - T(F)}{t} = \frac{d}{dt}T(F_t)|_{t=0} \equiv IC(x; T, F) \equiv \psi_F(x)
\]

\[
= \frac{d}{dt} \left\{ \int_0^{t_0} \frac{1}{1 - (F_t)_-} dF_t \right\} |_{t=0}
\]

\[
= \int_0^{t_0} \frac{1}{1 - F_-} d(\delta_x - F) + \int_0^{t_0} \frac{(\delta_x - F_-)}{(1 - F_-)^2} dF
\]

\[
= \frac{1_{[0,t_0]}(x)}{1 - F_-(x)} - \int_0^{t_0} \frac{1}{1 - F_-} dF + \int_0^{t_0} \frac{1}{1 - F_-} dF - \int_0^{t_0} \frac{(1 - \delta_x_-)}{(1 - F_-)^2} dF
\]

\[
= \frac{1_{[0,t_0]}(x)}{1 - F_-(x)} - \int_0^{t_0} \frac{1_{[x \geq y]}}{(1 - F_-)^2} dF(y)
\]

\[
= \frac{1_{[0,t_0]}(x)}{1 - F_-(x)} - \int_x^{t_0} \frac{1_{[0,t_0]}(y)}{1 - F_-} d\Lambda(y)
\]

\[
= \begin{cases} 
\frac{1_{1-F_-(x)}}{1-F_-(x)} - \int_0^{x} \frac{1}{(1-F_-)^2} dF & \text{if } x \leq t_0 \\
- \int_0^{t_0} \frac{1}{(1-F_-)^2} dF & \text{if } x > t_0
\end{cases}
\]

The next to last formula for the influence function of \(\Lambda(t_0)\) is natural from a martingale perspective. When \(F\) is continuous \(F_- = F\), and the influence function computed above reduces to:

\[
IC(x; T, F) = 1_{[x \leq t_0]} - \frac{F(t_0)}{1 - F(t_0)} 1_{[x > t_0]} = \frac{1_{[x \leq t_0]} - F(t_0)}{1 - F(t_0)}.
\]

Note that \(E_F \psi_F(X) = 0\) and (in the case of a continuous d.f. \(F\))

\[
E_F \psi_F^2(X) = \frac{F(t_0)}{1 - F(t_0)}.
\]

To prove asymptotic normality of \(T(F_n)\) (assuming that \(F\) satisfies \(F(t_0) < 1\),

\[
3
\]
\[ \sqrt{n}(T(F) - T(F)) = \sqrt{n} \left\{ \int_0^{t_0} \frac{1}{1 - F_n(s-)} dF_n(s) - \int_0^{t_0} \frac{1}{1 - F(s-)} dF(s) \right\} \]
\[ = \int_0^{t_0} \frac{1}{1 - F_n(s-)} d[\sqrt{n}(F_n(s) - F(s))] \]
\[ + \int_0^{t_0} \sqrt{n} \left\{ \frac{1}{1 - F_n(s-)} - \frac{1}{1 - F(s-)} \right\} dF(s) \]
\[ = \frac{\sqrt{n}(F_n(t_0) - F(t_0))}{1 - F_n(t_0-)} - \int_0^{t_0} \sqrt{n}(F_n(s) - F(s)) \frac{1}{(1 - F_n(s-))^2} dF_n(s) \]
\[ + \int_0^{t_0} \sqrt{n}(F_n(s) - F(s)) \frac{1}{(1 - F_n(s-))(1 - F(s-))} dF(s) \]
\[ = \frac{\sqrt{n}(F_n(t_0) - F(t_0))}{1 - F_n(t_0-)} + o_p(1) \]
\[ \to d \quad \frac{U(F(t_0))}{1 - F(t_0-)} \]
\[ \sim N(0, \frac{F(t_0)}{1 - F(t_0)}) \quad \text{if } F \text{ is continuous} \]

since the last two terms can be rewritten as
\[ \int_0^{t_0} \frac{\sqrt{n}(F_n(s) - F(s))}{1 - F_n(s-)} \left\{ \frac{dF_n(s)}{1 - F_n(s-)} - \frac{dF(s)}{1 - F(s-)} \right\} = o_p(1) \]

by arguments similar to those we used to deal with the Mann-Whitney Wilcoxon statistic. Alternatively, martingale methods also work.

2. Let \( T(F) \equiv \int (x - \mu(F))^3 dF(y)/\sigma^3(F) \) be the skewness functional where \( \mu(F) \equiv \int x dF(x) \) and \( \sigma^2(F) = \int (x - \mu(F))^2 dF(x) \).

(a) For what collection of df’s \( F_0 \) is \( T \) weakly continuous at \( F_0 \)? For what collection of df’s \( F_0 \) is \( T \) continuous at \( F_0 \) with respect to the Kolmogorov metric?

(b) Find the influence function of \( T(F) \).

Hint: First calculate the influence functions of \( \mu(F) \) and \( \sigma^2(F) \); then use the chain rule.

Comment: part (b) is problem 1, Wasserman, page 39; the influence function he gives for \( T \) on page 29 does not seem to be correct.

Solution: (a) Suppose that \( \mathcal{F}_{M,\delta} \equiv \{ F : E_F|X|^{3+\delta} \leq M \} \) for some \( 0 < M < \infty \) and \( \delta > 0 \). Then if \( \{ F_n \} \subset \mathcal{F}_{M,\delta} \) satisfies \( F_n \to_d F_0 \), it follows from Vitali’s
theorem that
\[(V_3(F_n), V_2(F_n), V_1(F_n)) \equiv \left( \int x^3 dF_n(x), \int x^2 dF_n(x), \int x dF_n(x) \right) \rightarrow \left( \int x^3 dF(x), \int x^2 dF(x), \int x dF(x) \right).\]

Slightly more generally this holds for any sequence \(\{F_n\}\) for which \(F_n \rightarrow_d F\) and \(|x|^3\) is \(F_n\)-uniformly integrable:
\[
\limsup_{n \rightarrow \infty} \int_{|x| \geq \lambda} |x|^3 dF_n(x) \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow \infty. \tag{1}
\]

Since \(T(F)\) is a continuous function of \((V_3(F), V_2(F), V_1(F))\) at all \(F\) such that \(\sigma^2(F) = V_2(F) - V_1^2(F) > 0\), it follows that \(T(F_n) \rightarrow T(F)\) for all \(\{F_n\}\) satisfying \(F_n \rightarrow F\) and (1) if \(\sigma^2(F) > 0\).

Since \(\|F_n - F\|_\infty \rightarrow 0\) implies that \(F_n \rightarrow_d F\), if the uniform integrability condition (1) holds and \(\sigma^2(F) > 0\), then \(T(F_n) \rightarrow T(F)\).

(b) To find the influence function of \(T(F) = \int (x - \mu)^3 dF(x) / \sigma(F)^3\), let \(F_t \equiv (1 - t)F + tG\). Then we need to compute \((d/dt)T(F_t)|_{t=0}\). But, by using the
calculations in examples 7.4.2 and 7.4.3,

\[
\frac{d}{dt} T(F_t)|_{t=0} = \frac{d}{dt} \left[ \frac{(x - \mu(F_t))^3 dF_t(x)}{[\sigma^2(F_t)]^{3/2}} \right]_{t=0}
\]

\[
= \left[ \frac{\int (x - \mu(F_t))^3 d(G - F)(x)}{[\sigma^2(F_t)]^{3/2}} \right]_{t=0}
\]

\[
- \frac{3}{2} \left[ \frac{\int (x - \mu(F_t))^3 dF_t(x) d \sigma^2(F_t)}{[\sigma^2(F_t)]^{5/2}} \right]_{t=0}
\]

\[
- \frac{3}{\sigma(F_t)^3} \left[ \frac{\int (x - \mu(F_t))^2 dF_t(x) d \mu(F_t)}{dt} \right]_{t=0}
\]

\[
= \int \left( \frac{x - \mu(F)}{\sigma(F)} \right)^3 d(G - F)(x)
\]

\[
- \frac{3}{2} T(F) \left\{ \int (x - \mu(F))^2 dG(x) - \sigma^2(F) \right\}
\]

\[
-3 \int \left( \frac{x - \mu(F)}{\sigma(F)} \right) dG(x)
\]

\[
= \int \left( \frac{x - \mu(F)}{\sigma(F)} \right)^3 dG(x) - T(F)
\]

\[
- \frac{3}{2} T(F) \int \left\{ \left( \frac{x - \mu(F)}{\sigma(F)} \right)^2 - 1 \right\} dG(x)
\]

\[
-3 \int \left( \frac{x - \mu(F)}{\sigma(F)} \right) dG(x)
\]

Hence by taking \( G = \delta_x \) we find the influence function of \( T(F) \):

\[
\hat{T}(F; \delta_x - F) = \left( \frac{x - \mu(F)}{\sigma(F)} \right)^3 - T(F) - \frac{3}{2} T(F) \left\{ \left( \frac{x - \mu(F)}{\sigma(F)} \right)^2 - 1 \right\}
\]

\[
-3 \left( \frac{x - \mu(F)}{\sigma(F)} \right)
\]

\[
\equiv \psi_F(x).
\]

Note that this derivation does not seem to agree with the result stated on page 29 of Wasserman: the third term here does not appear in Wasserman’s claimed influence function.
(b) Here is a direct calculation to see the result in (a) another way. Write
\[
\sqrt{n}(T(\mathbb{F}_n) - T(F)) = \frac{1}{\sigma(F)^3}\sqrt{n}\left\{ \int (x - \mu(\mathbb{F}_n))^3 d\mathbb{F}_n(x) - \int (x - \mu(F))^3 dF(x) \right\} + \int (x - \mu(F))^3 dF(x) \sqrt{n} \left\{ \frac{1}{\sigma(\mathbb{F}_n)^3} - \frac{1}{\sigma(F)^3} \right\}
\equiv A_n + B_n.
\]
To understand \( A_n \), write
\[
(x - \mu(\mathbb{F}_n))^3 = (x - \mu(F) - (\mu(\mathbb{F}_n) - \mu(F)))^3 \equiv (a - b)^3 = a^3 - 3a^2 b + 3ab^2 + b^3 = (x - \mu(F))^3 - 3(x - \mu(F))^2(\mu(\mathbb{F}_n) - \mu(F)) + 3(x - \mu(F))(\mu(\mathbb{F}_n) - \mu(F))^2 + (\mu(\mathbb{F}_n) - \mu(F))^3.
\]
Thus we see that
\[
A_n = \frac{1}{\sigma(F)^3} \left\{ \sqrt{n} \int (x - \mu(F))^3 d(\mathbb{F}_n(x) - F(x)) - 3 \int (x - \mu(F))^2 d\mathbb{F}_n(x) \sqrt{n}(\mu(\mathbb{F}_n) - \mu(F)) + 3 \int (x - \mu(F)) d\mathbb{F}_n(x) \sqrt{n}(\mu(\mathbb{F}_n) - \mu(F))^2 + \sqrt{n}(\mu(\mathbb{F}_n) - \mu(F))^3 \right\} + o_p(1)
\]
\[
= \frac{1}{\sigma(F)^3} \left\{ \sqrt{n} \int (x - \mu(F))^3 d(\mathbb{F}_n(x) - F(x)) - 3\sigma^2(F) \sqrt{n} \int (x - \mu(F)) d(\mathbb{F}_n(x) - F(x)) \right\} + o_p(1).
\]
For \( B_n \) we can write, with \( m_3(F) \equiv \int (x - \mu(F))^3 dF(x) \)
\[
B_n = m_3(F) \sqrt{n} \left\{ \frac{1}{\sigma(\mathbb{F}_n)^3} - \frac{1}{\sigma(F)^3} \right\}
\]
\[
= - \frac{m_3(F)}{\sigma(F)^3 \sigma(\mathbb{F}_n)^3} \sqrt{n} \left\{ \sigma(\mathbb{F}_n)^3 - \sigma(F)^3 \right\}
\]
\[
= - \frac{m_3(F)}{\sigma^2(F)^3} \sqrt{n} \left\{ \sigma^2(\mathbb{F}_n)^{3/2} - \sigma^2(F)^{3/2} \right\} + o_p(1)
\]
\[
= - \frac{m_3(F)}{\sigma^2(F)^3} \frac{3}{2} \sigma(F) \sqrt{n} (\sigma^2(\mathbb{F}_n) - \sigma^2(F)) + o_p(1)
\]
\[
= - \frac{m_3(F)}{\sigma^3(F)} \frac{3}{2} \sigma^2(F) \sqrt{n} \int \{(x - \mu(F))^2 - \sigma^2(F)\} d\mathbb{F}_n(x).
\]
\[
7
\]
Putting the $A_n$ and $B_n$ pieces together we see that we have complete agreement with the result of the influence function calculation:

$$\sqrt{n}(T(F_n) - T(F)) = \sqrt{n} \int \psi_F(x) dF_n(x) + o_p(1)$$

where $\psi_F(x)$ is as given in (2). It is clear (from the Central Limit Theorem) that this is asymptotically normal if $E_F X^6 < \infty$.

When I use the influence function derived here to obtain an estimator of the Standard Error of the skewness estimator for the nerve data treated in Wasserman’s example 3.10, page 29, I get $\hat{s} \hat{c} = .163$ rather than Wasserman’s estimate of .18, a slight reduction. The resulting confidence interval for the population skewness is $1.76 \pm 2(.163) = (1.434, 2.086)$.

3. Suppose that $\mathcal{F}_+$ is the class of distribution functions $F$ on $\mathbb{R}^+$ with mean $\mu_F = E_F X < \infty$, and consider the functional $T(F)$ defined for a fixed $x_0 \in \mathbb{R}^+$ by

$$T(F) \equiv e_F(x_0) \equiv E_F(X - x_0 | X > x_0) = \frac{\int_{x_0}^{\infty} (1 - F(t)) dt}{1 - F(x_0)}.$$

This functional is the mean residual life functional.

(a) For what collection of df’s $F_0$ is $T$ weakly continuous at $F_0$? For what collection of df’s $F_0$ is $T$ continuous at $F_0$ with respect to the Kolmogorov metric?

(b) Find the influence function of $T(F)$. (Consider expressing $T(F)$ in terms of two simpler functionals $U(F)$ and $V(F)$ and using the chain rule.)

**Solution:** (a) Much as in the previous problem we can regard $T(F)$ as the ratio of two simpler functionals: $T(F) = U(F)/V(F)$ where

$$U(F) \equiv E_F \{ (X - x_0) 1 \{ X > x_0 \} \} = \int_{x>x_0} (x-x_0) dF(x)$$

and $V(F) \equiv 1 - F(x_0) = \int_{x>x_0} dF(x)$. The second of these is continuous with respect to weak convergence at continuity points $x_0$ of $F_0$: $V(F_n) = 1 - F_n(x_0) \rightarrow 1 - F_0(x_0)$ if $x_0$ is a continuity point of $F_0$. On the other hand, $U(F)$ is weakly discontinuous at every $F_0$ if we allow arbitrary sequences $F_n$: taking $F_n(x) = (1 - n^{-1})F_0(x) + n^{-1} I_{[a_n,\infty)}(x)$, then

$$U(F_n) = (1 - n^{-1}V(F_0)) + n^{-1} (a_n - x_0) 1 \{ a_n > x_0 \} \rightarrow V(F_0) + \infty = \infty$$

if $a_n$ satisfies $n^{-1}a_n \rightarrow \infty$. But if we restrict to sequences $\{F_n\}$ such that $|x|-$ is $F_n$ uniformly integrable, then we have

$$U(F_n) = \int_{x>x_0} (x-x_0) dF_n(x) \rightarrow \int_{x>x_0} (x-x_0) dF_0(x) = U(F_0),$$
and we conclude that $T(F) = U(F)/V(F)$ is weakly continuous at $F_0$ with $V(F_0) = 1 - F_0(x_0) > 0$ with respect to all sequences $\{F_n\}$ that satisfy $F_n \to F_0$ and for which $|x|$ is $F_n$-uniformly integrable.

(b) First note that with $F_t \equiv (1 - t)F + tG$ we have both

$$
\frac{d}{dt}(1 - F_t(x_0))|_{t=0} = -(G - F)(x_0)
$$

and

$$
\frac{d}{dt} \int_{x_0}^{\infty} (1 - F_t(y))dy|_{t=0} = -\int_{x_0}^{\infty} (G - F)(y)dy.
$$

Thus by the product rule we calculate

$$
\frac{d}{dt} T(F_t)|_{t=0} = -\int_{x_0}^{\infty} (G - F)(y)dy \frac{1}{1 - F(x_0)} + \int_{x_0}^{\infty} (1 - F(y))dy \frac{1 - F(x_0)}{(1 - F(x_0))^2} (G - F)(x_0)
$$

$$
= e_F(x_0) (G - F)(x_0) \frac{1}{1 - F(x_0)} - \int_{x_0}^{\infty} (G - F)(y)dy \frac{1 - F(x_0)}{1 - F(x_0)}
$$

Taking $G = \delta_x = 1_{[x,\infty)}$ yields the influence function for $T$ at $F$:

$$
IC(x; T, F) = e_F(x_0) \frac{1_{[x,\infty)}(x_0) - F(x_0)}{1 - F(x_0)} - \int_{x_0}^{\infty} (1_{[x,\infty)}(y) - F(y))dy \frac{1}{1 - F(x_0)}
$$

$$
= e_F(x_0) \frac{1_{[0,x]}(x) - F(x_0)}{1 - F(x_0)} - \int_{x_0}^{\infty} (1_{[0,y]}(x) - F(y))dy \frac{1 - F(x_0)}{1 - F(x_0)}
$$

$$
= \begin{cases} 
  e_F(x_0) - \int_{0}^{\infty} 1_{[0,\infty)}(y)dy \frac{1 - F(x_0)}{1 - F(x_0)} & x \leq x_0 \\
  -F(x_0) - e_F(x_0) \frac{1}{1 - F(x_0)} - \int_{x_0}^{\infty} 1_{[0,\infty)}(y) - F(y)dy \frac{1}{1 - F(x_0)} & x > x_0 
\end{cases}
$$

$$
= \begin{cases} 
  0 & x \leq x_0 \\
  -F(x_0) - e_F(x_0) \frac{1}{1 - F(x_0)} - \int_{x_0}^{\infty} F(y)dy \frac{1}{1 - F(x_0)} & x > x_0 
\end{cases}
$$

$$
= \begin{cases} 
  0 & x \leq x_0 \\
  \frac{(x - x_0) - e_F(x_0)}{1 - F(x_0)} & x > x_0 
\end{cases}
$$

$$
= \frac{[x - x_0 - e_F(x_0)]1_{(x_0,\infty)}(x)}{1 - F(x_0)}.
$$

Note that

$$
E_F[IC^2(X; T, F)] = \frac{Var(X - x_0|X > x_0)}{1 - F(x_0)}.
$$