Due: Thursday, April 28, 2016

1. (a) The moment condition for (the extended form of) Bernstein’s inequality is:
   \[ E|X_i|^k \leq k!c^{k-2}v_i/2 \] for every \( k \geq 2 \) and all \( i \leq n \) and constants \( c > 0 \) and \( v_i \).
   Show that this holds if we have
   \[
   E \left( e^{X_i/c} - 1 - \frac{|X_i|}{c} \right) c^2 \leq \frac{1}{2} v_i.
   \]
   On the other hand, show that if the moment condition holds, then the previous display holds with \( c \) replaced by \( 2c \) and \( v_i \) replaced by \( 2v_i \).
   (b) Show that the moment condition of Bernstein’s inequality holds with
   \( c = M/3 \) if \( |X_i| \leq M \) with probability 1.

2. BLM, page 79, problem 3.7: Show that the conditional Rademacher average \( Z \)
satisfies the self-bounding property. Here \( Z \) is defined by
   \[
   Z \equiv E \left\{ \max_{1 \leq j \leq d} \sum_{i=1}^n \epsilon_i X_{i,j} | X_1, \ldots, X_n \right\}
   \]
   where \( X_1, \ldots, X_n \) are independent random variables taking values in \([-1, 1]^d\) and
   \( \epsilon_1, \ldots, \epsilon_n \) are independent Rademacher random variables which are independent of the \( X_i \)’s

3. BLM, page 78, problem 3.5: Consider the class \( F \) of functions \( f : \mathbb{R}^n \to \mathbb{R} \) that
   are Lipschitz with respect to the \( \ell^1 \) distance: i.e.
   \[
   |f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n)| \leq \sum_{i=1}^n |x_i - y_i|.
   \]
   Let \( X = (X_1, \ldots, X_n) \) be a vector of independent random variables with finite variance.
   Use the Efron - Stein inequality to show that the maximal value of
   \( \text{Var}(f(X)) \) over \( f \in F \) is attained by the function \( f(x) = \sum_{i=1}^n x_i \).
   (This is from Bobkov and Houdré (1996).)

4. BLM, page 114, problem 4.11: prove that for any fixed probability measure \( P \) on \( \mathcal{X} \), the function \( Q \mapsto D(Q \| P) \) is convex on the set of probability distributions
   over \( \mathcal{X} \). Hint: Use the duality representation.
5. BLM, page 114, problem 4.13: Let $Z$ be a real-valued random variable. Recall that $\psi_Z(\lambda) = \log E e^{\lambda Z}$ for $\lambda \in \mathbb{R}$. Let $\psi^*(t) = \sup_{\lambda \in \mathbb{R}} \{\lambda t - \psi_{Z-E(Z)}(\lambda)\}$. Prove that for all $t > 0$

$$\psi^*(t) = \inf \{D(Q\|P) : E_Q(Z) - E(Z) \geq t\}.$$ 

6. **Bonus problem:** BLM, page 115, problem 4.17: Let $C$ be a convex body (a compact convex set with nonempty interior) in $\mathbb{R}^n$, and let $P$ be the uniform probability distribution over $C$. Prove Borell’s lemma that states the following: if $A$ is a symmetric convex subset of $C$ with $P(A) > 1/2$, then for any $t > 1$,

$$P((tA)^c) \leq P(A) \left(\frac{1 - P(A)}{P(A)}\right)^{(t+1)/2}.$$ 

**Hint:** Prove first that for $t > 1$

$$\frac{2}{t+1} (tA)^c + \frac{t-1}{t+1} A \subset A^c$$

where $A^c = C \setminus A = C \cap A^c$ where the complement on the right side is the usual complement in $\mathbb{R}^n$. Then use the Brunn-Minkowski inequality. (This is an example of the concentration of measure phenomenon: note that the inequality does not depend on the ambient dimension $n$.)