Due: Thursday, April 14, 2016

1. BLM, page 47, problem 2.7: Prove that if $Z$ is a centered normal random variable with variance $\sigma^2$, then

$$P(Z \geq t) \leq \frac{1}{2} \exp\left(-\frac{t^2}{2\sigma^2}\right) \text{ for all } t > 0.$$ 

Solution: Without loss of generality let $\sigma^2 = 1$; if not, consider $Z/\sigma \sim N(0,1)$. Now the claimed inequality holds if and only if

$$\int_t^\infty \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)dy \leq 1/2 \text{ for all } t > 0.$$ 

But by letting $z = y - t, y = z + t$, and hence $y + t = z + 2t$, the integral on the left side in the last display can be rewritten as

$$\int_0^\infty \frac{1}{\sqrt{2\pi}} \exp(-z(z+2t)/2)dz = \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)\exp(-zt)dz \leq \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)dz = 1/2.$$ 

2. BLM, page 47, problem 2.7: Elementary inequalities:

$$-\log(1-u) - u \leq \frac{u^2}{2(1-u)} \text{ for } u \in (0,1);$$

$$\tilde{h}(u) = (1+u)\log(1+u) - u \geq \frac{u^2}{2(1+u/3)} \text{ for } u > 0;$$

$$h_1(u) = 1 + u - \sqrt{1+2u} \geq \frac{u^2}{2(1+u)}, \text{ for } u > 0.$$ 

Solution: (i) The claimed inequality, $-\log(1-u) - u \leq u^2/(2(1-u))$ holds for $0 \leq u < 1$ if and only if

$$-(1-u)\log(1-u) - u(1-u) \leq u^2/2, \quad 0 \leq u < 1,$$
which holds if and only if 
\[-(1 - u) \log(1 - u) - u + u^2/2 \leq 0, \quad \text{for } 0 \leq u < 1.\]

Let \( H(u) \equiv -(1 - u) \log(1 - u) - u + u^2/2, \) the expression on the left side in the last display. Then \( H(0) = 0 \) and

\[ H'(u) = \log(1 - u) + u \leq -u + u = 0, \quad 0 \leq u < 1. \]

Hence \( H(u) \leq 0. \)

(ii) The claimed inequality is

\[ \overline{h}(u) = (1 + u) \log(1 + u) - u \geq \frac{u^2}{2(1 + u/3)} \equiv g(u) \]

Note that \( g(u) = 3u^2/(2(3 + u)). \) Thus we have \( \overline{h}(0) = g(0) = 0 \) and we have

\[ \overline{h}'(u) = 1 + \log(1 + u) - 1 = \log(1 + u), \]

\[ g'(u) = \frac{6u}{2(3 + u)} - \frac{3u^2}{2(3 + u)^2} = \frac{3u(6 + u)}{2(3 + u)^2}, \]

so that \( \overline{h}'(0) = 0 = g'(0). \) Furthermore, \( g''(u) = 27/(3 + u)^3 \leq 1/(1 + u) = \overline{h}''(u). \) Integration of this latter inequality twice yields the result.

(iii) The third inequality holds if \( 1 + u - \sqrt{1 + 2u} \geq u^2/(2(1 + u)) \); equivalently, this holds if and only if

\[ (1 + u)^2 - (1 + u)(1 + 2u)^{1/2} \geq u^2/2; \]

or if and only if

\[ 1 + 2u + \frac{u^2}{2} \geq (1 + u)(1 + 2u)^{1/2}. \]

Squaring both sides we see that this holds if and only if

\[ (1 + 2u + \frac{u^2}{2})^2 \geq (1 + u)^2(1 + 2u). \]

Here the right side equals

\[ 1 + 4u + 5u^2 + 2u^3, \]

while the left side equals

\[ 1 + 4u + 5u^2 + 2u^3 + \frac{u^4}{4}. \]

Thus the claimed inequality holds.
3. Verify the elementary calculation required to show that

\[
\psi^*(t) \equiv \sup_{0<\lambda<1/c} \left( t\lambda - \frac{\nu\lambda^2}{2(1-c\lambda)} \right) = \frac{\nu}{c^2} h_1 \left( \frac{ct}{\nu} \right)
\]

where \( h_1(u) = 1 + u - \sqrt{1 + 2u}, \ u > 0. \)

**Solution:** Let \( x \equiv c\lambda \) and \( s = ct/\nu. \) Then write

\[
t\lambda - \frac{\nu\lambda^2}{2(1-c\lambda)} = \frac{\nu}{c^2} \left\{ tc\lambda - \frac{c^2\lambda^2}{2(1-c\lambda)} \right\} = \frac{\nu}{c^2} \left\{ sx - \frac{x^2}{2(1-x)} \right\} \equiv \left( \frac{\nu}{c^2} \right) h_s(x)
\]

Computing \( h'_s \) yields

\[
h'_s(x) = \frac{1}{2(1-x)^2} \left\{ 2s - 2(1+2s)x + (1+2s)x^2 \right\}.
\]

Then the right root of \( h'_s(x) \) is \( x^* = 1 - 1/\sqrt{1+2s}, \) and then careful computation yields

\[
h_s(x^*) = 1 + s - \sqrt{1+2s} = h_1(s)
\]

as claimed.

4. BLM, page 49, problem 2.18: (Maximum of independent Poisson random variables). Let \( X_1, \ldots, X_n \) be independent Poisson random variables with expectation 1. The Lambert \( W \) function is defined over \([-1/e, \infty)\) by the equation \( W(x)e^{W(x)} = x. \)

(a) Prove that

\[
E \max_{1 \leq i \leq n} X_i \leq \frac{\log(n/e)}{W(\log(n/e)/e)}.
\]

(1)

(b) Prove that for \( z \geq e, \) \( W(z) \geq \log(z) - \log \log(z) \) and that for \( n \geq e^3, \)

\[
E \max_{1 \leq i \leq n} X_i \leq \frac{\log(n/e)}{\log(\log(n/e)/e) - \log(\log(\log(n/e)/e))}.
\]

(2)

The following upper bound may be more manageable:

\[
E \max_{1 \leq i \leq n} X_i \leq \frac{2\log n}{\log(\log(en))}.
\]

(3)
Remarks: (i) I can only prove the claim of (b) for \( n \geq e^{1+e^2} \), rather than for \( n \geq e^3 \) as claimed.
(ii) I believe that the bound (3) should be modified to
\[
E \max_{1 \leq i \leq n} X_i \leq \frac{2 \log(n/e)}{\log \log(n/e)} \quad \text{if} \quad n \geq e^2.
\]
(iii) Note that independence is not used.

Solution: It is easily seen from the Poisson example on page 23 of BLM that for \( X_i \)'s distributed as Poisson(\( \lambda \)), \( Z_i \equiv X_i - \lambda \) have log-MGF
\[
\psi^*(s) = \lambda \overline{h}(s/\lambda) \quad \text{if} \quad s \geq 0
\]
and 0 otherwise. Here \( \overline{h}(v) = h(1+v) \) where \( h(x) \equiv x(\log x - 1) + 1 \) and hence \( \overline{h}(0) = \overline{h}'(0) = 0 \). By BLM Theorem 2.5 we have
\[
E \max_{1 \leq i \leq N} Z_i \leq \psi^{-1}_* (\log N);
\]
note that this apparently holds even if the \( Z_i = X_i - \lambda \)'s are not independent. It remains only to show that when \( \lambda = 1 \)
\[
\psi^{-1}_*(y) = \overline{h}^{-1}(y) = e^{(y-1)/e} W((y-1)/e) - 1. \tag{4}
\]
where \( W \) satisfies \( W(x) \exp(W(x)) = x \). To see this, note that when \( \lambda = 1 \) we just need to show that \( \overline{h}(\psi^{-1}_*(y)) = y \). But if \( \psi^{-1}_* \) is as in the last display we have, since \( h(x) = x(\log x - 1) + 1 \) and \( \overline{h}(v) = h(1+v) \),
\[
\overline{h}(\psi^{-1}_*(y)) = h\left(e^{(y-1)/e} W((y-1)/e)\right)
= \frac{e^{(y-1)/e}}{W((y-1)/e)} \left( \log \left( \frac{(y-1)/e}{W((y-1)/e)} \right) + 1 - 1 \right) + 1
= \frac{e^{(y-1)/e}}{W((y-1)/e)} W((y-1)/e) + 1 \quad \text{since} \quad e^{W(x)} = \frac{x}{W(x)}
= y - 1 + 1 = y.
\]
Thus (4) holds and hence also the first bound claimed above holds:
\[
E \max_{1 \leq i \leq N} (X_i - 1) = E \max_{1 \leq i \leq N} Z_i \leq \psi^{-1}_*(\log N) = e^{(\log N - 1)/e} W((\log N - 1)/e) - 1,
\]
and hence also
\[
E \max_{1 \leq i \leq N} X_i \leq \frac{\log N - 1}{W((\log N - 1)/e)} = \frac{\log(N/e)}{W(\log(N/e)/e)}.
\]
To show that $W(z) \geq W_0(z) \equiv \log z - \log \log z$, note that

$$W_0(z) \exp(W_0(z)) = z \left(1 - \frac{\log \log z}{\log z}\right) \leq z = W(z) \exp(W(z))$$

for $z \geq e$. Since $y \mapsto ye^y$ is monotone increasing, this implies that $W_0(z) \leq W(z)$. Taking $z = \log(N/e)/e$ and noting that $z \geq e$ if $N \geq e^{(1+e^2)} \approx 4399$ yields the second bound: if $N \geq e^{(1+e^2)}$ then

$$E \max_{1 \leq i \leq N} X_i \leq \frac{\log(N/e)}{W(\log(N/e)/e) - \log(\log(\log(N/e)/e))}.$$

Note that $e^{(1+e^2)} \approx 4399 > 20.0855 \approx e^3!$

For the last claim, note that $W(z)/\log(\log(z)) \geq 1/2$ for all $z \geq 1/e$ (proof below), and hence $W(\log(N/e)/e) \geq (1/2) \log(\log(N/e))$ for all $N$ such that $\log(N/e)/e \geq 1/e$, or, equivalently for $N \geq e^2 \approx 7.38906$. Thus we have

$$E \max_{1 \leq i \leq N} X_i \leq \frac{\log(N/e)}{W(\log(N/e)/e) - \log(\log(N/e)/e)} \leq 2 \frac{\log(N/e)}{\log(\log(N/e))}.$$

for $N \geq e^2$.

**Proof of the claim that** $W(z)/\log(\log(z)) \geq 1/2$ **for all** $z \geq 1/e$

The claim can be rewritten as $W(z) \geq (1/2) \log(\log(z))$ for $z \geq 1/e$. Since $W$ is monotone non-decreasing for $z \geq z_0$, this is equivalent to

$$z = W(z)e^{W(z)} \geq (1/2) \log(\log(z)) \exp((1/2) \log(\log(z))) = (ez)^{1/2} \log ((ez)^{1/2}) \equiv y \log y$$

where $y \equiv (ez)^{1/2}$. But the last display is then equivalent to $y^2/e \geq y \log y$ for $y \geq 1$, or $g(y) \equiv y^2 - ey \log y \geq 0$ for all $y \geq 1$. Now $g(1) = 1$ and we find that $g'(y) = 2y - e - e \log y$ has $g'(1) = 2 - e < 0$ and $g'(e) = 0$ while $g'(y) > 0$ for $y > e$ with $g''(y) = 2 - e/y$ has $g''(e) = 2 - e/e = 1 > 0$. Thus $g(y) \geq 0$ for all $y \geq 1$, and this yields the claim.

If we assume that the $X_i$’s are independent, then the bound of van der Vaart and W (1996), problem 2.3.5, page 121, becomes available: with $r = 1$ that upper bound becomes, for i.i.d. $X_i$’s and any $t_0 > 0$

$$E\{\max_{1 \leq i \leq N} X_i\} \leq t_0 + N \int_{t_0}^{\infty} P(X_1 > t) dt.$$  (5)

Minimizing this bound with respect to $t_0$ we find that the minimizing $t_0$ satisfies $P(X_1 > t_0) = 1/N$; i.e. $t_0 = F_1^{-1}(1 - 1/N)$ when the $X_i$’s are Poisson($\lambda$). The dots in Figure 2 represent the averages of 1000 Monte Carlo replications of the maximum of independent Poisson(1) random variables. The magenta curve in
Figure 2 gives the first bound in BLM Exercise 2.18, while the red dots give the bound in (5) with $t_0 = F_1^{-1}(1 - 1/N)$. As can be seen, the latter bound based on independence gives a noticeable improvement over the bound for the maximum of arbitrarily dependent Poisson(1) random variables.
Figure 1: Comparison of the three bounds for the maximum of Poisson(1) random variables. In sub-figure 1a, the sample size $N$ ranges from 10 to 1000. The first bound is plotted in magenta, while the third bound is plotted in green. As can be seen from this part of the Figure, the second bound, plotted here in blue, is not valid for this range of $N$. In sub-figure 1b, we plot the three bounds for values of $N$ between 3000 and $10^4$. Note that the second bound, plotted again in blue, becomes valid at $N = 4399$. Also note that the third bound (in green) is an extremely accurate approximation for the first bound (in magenta) for $N$ in this range.