MONTE CARLO OF TWO-DIMENSIONAL BROWNIAN SHEETS

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1. Introduction. A Brownian sheet $Z(\mathbf{t})$ with an $r$-dimensional parameter set is a mean zero Gaussian process defined on the positive orthant of $r$-dimensional Euclidean space, $\mathbb{R}^+_r$, having covariance $E Z(\mathbf{t})Z(\mathbf{t}') = |\mathbf{t} \wedge \mathbf{t}'|$ (the minimum is taken coordinate by coordinate and $|\mathbf{t}'| = t_1 \ldots t_r$). Such processes arise, for example, as the (weak) limit of "partial sum processes" on an $r$-dimensional grid, $\mathbb{N}^+_r$. These processes have received increasing attention (see Pyke [3] for a recent review of the status of research), but few researchers have "seen" a Brownian sheet, and knowledge concerning the fluctuation theory of these processes is extremely scant. For example, the distributions of $M_\mathbf{t} = \sup_{0 \leq \mathbf{t} \leq \lambda} Z(\mathbf{t})$ and $T_\mathbf{t} = \lambda \{ \mathbf{t} \in \lambda; Z(\mathbf{t}) > 0 \}$ ($\lambda$ denotes Lebesgue measure on $\mathbb{R}^+_r$) are unknown for $r \geq 2$. (For $r = 1$ the distributions are, of course, well known: $M_1$ has a half-normal distribution and $T_1$ has the arcsin distribution.)

Here we present the results of a Monte Carlo experiment in which approximations to Brownian sheets on $\mathbb{R}^+_2$ were obtained as partial sums of Normal $(0,1)$ rv's. The aims of this experiment were
twofold: first, to plot several realizations of the
approximating partial sum process so that we could
"see" several Brownian sheets; second, to learn
about the distributions of \( M_2 \) and \( T_2 \) by examining
the empirical distribution functions of the
corresponding rv's defined on approximating partial
sum processes. A plot of the partial sum process
itself is shown in Section 2 and may give some
idea of the appearance of a Brownian sheet. In
Section 3 we present the empirical distributions
which estimate the distributions of \( M_2 \) and \( T_2 \). We
also show that the arcsin law does not hold for
\( r \geq 2 \); in fact, \( T_r \sim 1/2 \) as \( r \to \infty \). Finally, for the
sake of completeness, the Appendix lists the
computer programs used to generate the partial sum
processes and perform the computation of empirical
distributions.

The first simulations on Brownian sheets known
to this author were carried out at the University of
Minnesota in the summer of 1973, as communicated to
me through Professor John B. Walsh. The print-outs
in this case were "pictures" of the regions where
a Brownian sheet is positive. Similar pictures were
later obtained by Professor Pyke at the University
of Washington. A question raised by Professor Walsh
concerns the apparent rectangularity of the level
lines present in some of the \( \omega \)'s.

2. A picture of a Brownian sheet on \( \mathbb{R}^+_2 \). Figure 1
is a "three dimensional" plot of a sample path
(one \( \omega \) in the sample space, of a partial sum process
\( S_k \), \( 1 \leq k \leq n \) obtained by summing independent Normal
(0,1) rv's, $X_i$, as follows:

$$S_k = \sum_{i=1}^{k} X_i, \quad 1 \leq k \leq m.$$  \hfill (1)

The grid is 50 by 50 ($m = (50,50)$) and the sheet is viewed from the point $(x,y,z) = (-833,-833,833)$ in grid units. (Or from (-100,-100,+100) in inches with the 50 by 50 grid on a six inch square.) The origin is not plotted, so the entire sheet is seen relative to the value of $X_{\frac{1}{2}} = S_0$, and not $S_{\frac{1}{2}} = 0$.

Although the sheet appears to be entirely positive in this view, there are several valleys and "holes" which are not visible from this viewing position, but can be seen clearly from other angles. For a 25 by 25 grid, several sheets were viewed from all eight corners.

3. **Empirical distributions of $M^d_2$ and $T^d_2$.** In terms of the partial sums $S_k$, the random variables corresponding to $M^d_2$ and $T^d_2$ are

$$M^d_2 = \max_{1 \leq k \leq m} S_k / |m|^{\frac{1}{2}}$$  \hfill (2)

and

$$T^d_2 = \#\{k \leq m: S_k > 0\} / |m|.$$  \hfill (3)

Figures 2 through 5 present empirical distributions of $M^d_2$ and $T^d_2$ based on a 50 by 50 grid ($m = (50,50)$).
and two independent samples, each of size \( n = 400 \). (The empirical df of \( |m_1 - m_2| = \max_{k} S_k \) is plotted in Figures 4 and 5 rather than \( M_2^d \) itself.) Figures 6 and 7 show the locations of the maxima (K such that \( M_2^d = S_K \)) on the 50 by 50 grid for the same two samples. Each sample of 400 sheets, involving the generation of \( 10^5 \) normal random variables, required approximately 6 minutes of central processor time on the CDC 6400 at the University of Washington. The program used is given in an Appendix.

With only minor modifications the program could be used to obtain more detailed information concerning the distributions of functionals of the process \( Z \) with \( r = 2 \). For example, in view of the intractability of the distribution of \( M_2 \) (and its "tied down" analogue which is of importance for statistical applications), accurate estimates of several upper percent points of the distribution of \( M_2 \) would be of interest. Accurate estimates of quantiles of \( M_2 \) could be obtained using the present program with increased sample size, and estimates of quantiles for the "tied down" version of \( M_2 \) could be obtained by suitably transforming the partial sum process.

The empirical distributions of Figures 2 and 3 suggested that \( T_2 \) might have a uniform (0,1) distribution which could be made compatible with the 1-dimensional arcsin distribution through a Beta family of distributions. This null hypothesis that the distribution was uniform (0,1) was then tested using the two-sided Kolmogorov test. The hypothesis was accepted! However, the distribution
is not uniform \( (0,1) \) as may be seen by calculating the variance of \( T_2 \) using Sheppard's formula ([1], page 125) and doing some definite integrals one obtains

\[
\text{var } T_2 = (2\pi)^{-1} \int_0^1 \int_0^1 \arcsin \left( x_1 x_2 \right)^{\frac{1}{2}} \, dx_1 \, dx_2
\]

\[
= \left( \frac{1}{4} \right) \left( 1 - \log 2 \right).
\]

On the other hand, Figures 2 and 3 do suggest that the density function of \( T_2 \) is nearly flat, a dramatic change from the bowl-shaped \( \arcsin \) density of \( T_1 \), and leads one to the natural conjecture that the densities of \( T_r \) "turn around" as \( r \) increases and become concentrated about \( \frac{1}{2} \). The following proposition confirms this guess.

**Proposition.** \( T_r \sim \frac{1}{2} \) as \( r \to \infty \). (In fact, \( \text{var } T_r \leq \left( \frac{1}{4} \right) \left( \frac{2}{3} \right)^r \).)

**Proof.** Using Sheppard's formula and the inequality

\[
\arcsin (t) \leq (\pi/2)t \text{ for } 0 \leq t \leq 1
\]

we obtain

\[
\text{var } T_r = (2\pi)^{-1} \int_0^1 \int_0^1 \arcsin \left( |x|^\frac{1}{2} \right) |dx|
\]

\[
\leq \left( \frac{1}{4} \right) \int_0^1 \frac{1}{2} |x|^\frac{1}{2} |dx|
\]

\[
= \left( \frac{1}{4} \right) \left( \frac{2}{3} \right)^r
\]

as \( r \to \infty \)

and the conclusion follows since \( E T_r = \frac{1}{2} \) for all \( r \).

**Questions:** (1) Does \( T_r \), properly normalized, have a non-degenerate limiting distribution as \( r \to \infty \)? (2) What happens to the distribution of \( M_r \).
as $r \to \infty$? (Possibly an extreme value distribution?)

(3) The central question remains: Is there a "fluctuation theory" for $r$-dimensional sheets and arrays of which the known results for $r = 1$ (see [2], page 419) are a special case?

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REFERENCES


TWO-DIMENSIONAL BROWNIAN SHEETS

Figure 1. A Brownian sheet on $\mathbb{R}_2^+$. 
Figure 2. Empirical distribution of $T_2^a$, $n = 400$, sample 1.
Figure 3. Empirical distribution of $T_2^d$, $n = 400$, sample 2.
Figure 4. Empirical distribution of $M_2$, $n = 400$, sample 1.
Figure 5. Empirical distribution of $W^2_2$, $n = 400$, sample 2.
Figure 6. Location of the maxima, $K$, $n = 400$, sample 1.
Figure 7. Location of the maxima, $k$, $n = 400$, sample 2.
PROGRAM R4TWO (INPUT, OUTPUT, TAPE1=INPUT, TAPE2=OUTPUT, TAPE3=OUTPUT, TAPE4=OUTPUT)
DIMENSION L(400), SUP(400), PROP(400), ORD(400), ORD(21400), SUP(400)
DIMENSION Z(500), ORD(500), RL(400), RLJ(400)
COMMON /A/ L(400), LJ(400)
N = 10
DUM = RANF(17, 83561)
DO 1 L = 1, N
CALL SHIFT (SUP(L), PROP(L), L, Z)
1 CONTINUE
WRITE (6, 102) (SUP(J), PROP(J), LT(J), LJ(J), J = 1, N)
WRITE (6, 103) (Z(J), J = 1, N)
CALL ORDER (N, SUP)
CALL ORDER (500, Z)
RN = N
NG 2 L = 1, N
RL = L
ORD(L) = 81/N
2 CONTINUE
DO 4 J = 1, 500
RJ = J
ORD(J) = .0002*RJ
4 CONTINUE
101 FORMAT (1X+7OF12.4)
102 FORMAT (1X+7F12.4, 7I6)
103 FORMAT (1X+7F12.4)
104 FORMAT (1X+7I10)
WRITE (6, 105) (SUP(J), ORD(J), J = 1, N)
WRITE (6, 106) (PROP(J), ORD(J), J = 1, N)
WRITE (6, 107) (LT(J), LJ(J), J = 1, N)
WRITE (6, 108)
CALL PLOT1 (0.0, 25.0, 0.0, 1.0, 0.0)
CALL PLOT2 (0.0, 25.0, 0.0, 1.0, 0.0)
CALL PLOT3 (1.0, 0.0, 1.0, 0.0)
CALL PLOT4 (1.0, 0.0, 1.0, 0.0)
CALL PLOT5 (1.0, 0.0, 1.0, 0.0)
CALL PLOT3 (1.0, 0.0, 1.0, 0.0)
105 FORMAT (1X+7F12.4)
106 FORMAT (1X+7F12.4, 7I6)
107 FORMAT (1X+7F12.4)
108 FORMAT (1X+7F12.4, 7I6)
109 FORMAT (1X+7F12.4, 7I6)
110 FORMAT (1X+7F12.4, 7I6)
111 FORMAT (1X+7F12.4, 7I6)
112 FORMAT (1X+7F12.4, 7I6)
113 FORMAT (1X+7F12.4, 7I6)
114 FORMAT (1X+7F12.4, 7I6)
115 FORMAT (1X+7F12.4, 7I6)
116 FORMAT (1X+7F12.4, 7I6)
117 FORMAT (1X+7F12.4, 7I6)
118 FORMAT (1X+7F12.4, 7I6)
119 FORMAT (1X+7F12.4, 7I6)
120 FORMAT (1X+7F12.4, 7I6)
CALL PLOT4(11;11;11;PROBABILITY)
WRITE(6,101)
DO 49 J = 1,N
RLT(J) = LT(J) - RLJ(J) = L(J)
39 CONTINUE
CALL PLOT3(0;9;10;5;16)
CALL PLOT2(MAPC;50;0;5;50.0;5U;5U)
CALL PLOT3(1H4;RLJ;RLJ;N)
CALL PLOT4(12;12HY OR T2 AXIS)
WRITE(6,101)
PB = 0.0
DO 49 L = 1,N
T = 466.25/RLJ(L)
T[(RLJ(L)+GR,T)] PR = PR + 1.0
49 CONTINUE
PB = PB/RN
WRITE(6,107) PR
107 FORMAT(1x,#THE PROPORTION IN THE UPPER RIGHT CORNER 15.*E10.6)
DO 47 J = 1,N
WRITE(8*,14R) SUP(J),PRO(J),RL(J),RLJ(J),ORD(J)
14R FORMAT(1x,E10.6)
47 CONTINUE
STOP
END

SUBROUTINE NORMAL
COMMON S150,X1(50)
DO 2 J = 1,50
2 A = RAMP(0.0)
B = RAMP(0.0)
U = 2.*A - 1.
V = 2.*U - 1.
R2 = U*U + V*V
R = SORT(R2)
IF(R.GE.1.*) GO TO 2
RLOG = ALOG(R2)
S = SORT1(2,RLOG)
X(J+1) = U+S/R
X(J+1) = V*S/R
9 CONTINUE
RETURN
END
S U M R O U T I N E _ S W E E T ( M A X * P P + L + Z )
COMMON S (50, 15), X (40, 40)
COMMON A / L (400), L J (400)
D I M E N S I O N Z (400)
R E A L M A X , P P
D O 8 I = 1, M A X
D O 4 J = 1, P P
S (I + J) = 0.0
C O N T I N U E
C A L L N O R M A L
K K = -1
I F (L . G T . 1) G O T O 1 5
D O 2 9 J = 1, 5 0 0
Z (J) = 0.0
C O N T I N U E
D O 4 J = 1, 5 0
D O 9 J = 1, 5 0 0 + 1
K K = K K + 2
Z (K K) = X (I + J)
X (J) = X (I + J + 1)
C O N T I N U E
P P = 0.0
S (I + 1) = X (I + 1)
M A X = S (I + 1)
L I ( I ) = 1 0 5  L J ( I ) = 1
D O 7 J = 2, 5 0
S ( I , J ) = S ( I , J - 1 ) + X ( I + J)
S ( J , 1 ) = S ( J , 1 ) + X ( J + 1)
I F ( S ( I , J ) * L T . M A X ) G O T O 1 7
M A X = S ( I , J + 1)
L I ( I ) = J
I F ( S ( I + 1 , J ) * L T . M A X ) G O T O 1 7
M A X = S ( I + 1 , J)
L J ( I ) = J
1 3 I F ( S ( I + 1 , J ) * G T . 0 . 0 ) P P = P P + 1.0
I F ( S ( I + 1 , J ) * G T . 0 . 0 ) P P = P P + 1.0
C O N T I N U E
D O 8 I = 2, 5 0
D O 9 J = 2, 5 0
S ( I , J ) = X ( I + J ) + S ( I - 1 , J ) + S ( I , J - 1 ) - S ( I - 1 , J - 1 )
I F ( S ( I , J ) * L T . M A X ) G O T O 1 7
M A X = S ( I , J )
L I ( I ) = I 0 5  L J ( I ) = J
1 7 I F ( S ( I , J ) * G T . 0 . 0 ) P P = P P + 1.0
C O N T I N U E
P P = 0.0004 * P P
R E T U R N
E N D
SUBROUTINE ORDER(KK,Y)
DIMENSION Y(500)
M = KK
3 M = M/2
IF(M.LE.0) GO TO 8
K = KK-M
J = 1
4 T = J
5 IP = J + M
6 T = Y(I1)
Y(I1) = Y(IP)
Y(IP) = T
I = I - M
7 IF(I.EQ.1) THEN 7,5,5
8 RETURN
END

\[ \text{Var} \left( T_2 \right) = \frac{1}{4} \left( 1 - \log 2 \right) \]
\[ \mathbb{E} \left( T_2^2 \right) = \frac{1}{4} \left( 1 - \log 2 \right) + \left( \frac{1}{2} \right)^2 \]
\[ = \frac{1}{2} - \frac{1}{4} \log 2 \]
If $X$ has density $f_X(x) = \frac{1}{\pi \sqrt{x(1-x)}}$, then

$$
Y_1 = \frac{e^{itx} - 1}{x}
$$

where

$$
J_0(\frac{s}{2}) = \sum_{k=0}^{\infty} \frac{(-1)^k ((\frac{s}{2})^k}{k! k!}
$$

$$
J_0'(\frac{s}{2}) = \sum_{k=0}^{\infty} \frac{(-1)^k ((\frac{s}{2})^{2k}}{k! k!}
$$

$$
J_0''(\frac{s}{2}) = \sum_{k=0}^{\infty} \frac{(-1)^k ((\frac{s}{2})^{2k-2}}{k! k!}
$$


This paper contains a Monte Carlo study of the realizations and empirical distributions (of two functionals) of the two-dimensional Brownian sheet processes based on the approximation of the latter by partial sums of normal deviates.

{For the entire collection see MR 51 #1893.}

P. K. Sen (Chapel Hill, N. C.)