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# An excursion approach to maxima of the Brownian bridge

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## Abstract

Distributions of functionals of Brownian bridge arise as limiting distributions in non-parametric statistics. In this paper we will give a derivation of distributions of extrema of the Brownian bridge based on excursion theory for Brownian motion. The idea of rescaling and conditioning on the local time has been used widely in the literature. In this paper it is used to give a unified derivation of a number of known distributions, and a few new ones. Particular cases of calculations include the distribution of the Kolmogorov–Smirnov statistic and the Kuiper statistic.

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## 1. Introduction

Distributions of functionals of Brownian bridge arise as limiting distributions in non-parametric statistics. The distribution of the maximum of the absolute value of a Brownian bridge

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is the basis for the Kolmogorov–Smirnov non-parametric test of goodness of fit to give one example. For an overview of statistical applications see [23].

Let  $(\mathbb{U}_t : 0 \leq t \leq 1)$  be the standard Brownian bridge and define

$$M^+ = \max_{0 \leq t \leq 1} \mathbb{U}_t, \quad M^- = - \min_{0 \leq t \leq 1} \mathbb{U}_t \tag{1.1}$$

and

$$m = \min\{M^+, M^-\} \quad \text{and} \quad M = \max\{M^+, M^-\}. \tag{1.2}$$

The derivation of the distribution of  $M$  was given by Kolmogorov [12]. For an elementary derivation see [9]. In this paper we give a derivation of the joint distribution of  $M^+$  and  $M^-$  based on excursion theory for Brownian motion. The distributional results can be used to derive known distributions like the distribution of the Kuiper statistic  $K = M^+ + M^-$ , or the distribution of the difference  $D = M^+ - M^-$  which seems to be new.

Let  $(B_t : t \geq 0)$  be standard Brownian motion. Define the last exit time from 0 of  $B$  before time  $t$  as

$$g_t = \sup\{s \leq t : B_s = 0\}. \tag{1.3}$$

The following lemma is well known and will be used to derive distributional equalities needed later. See [15,7,1].

**Theorem 1.** *The distribution of  $g_1$  is Beta(1/2, 1/2). Given  $g_1$ , the process  $(B_t : 0 \leq t \leq g_1)$  is a Brownian bridge of length  $g_1$ , and the rescaled process  $(B(g_1u)/\sqrt{g_1} : 0 \leq u \leq 1)$  is a Brownian bridge independent of  $g_1$ .*

Let  $S_\theta \sim \exp(\theta)$  be independent from  $B$ . From scaling properties of Brownian motion and Theorem 1 it follows that the process

$$\left( \frac{B_{tg_{S_\theta}}}{\sqrt{g_{S_\theta}}} : 0 \leq t \leq 1 \right) \tag{1.4}$$

is a Brownian bridge independent of  $g_{S_\theta}$ . Furthermore, the law of  $g_{S_\theta}$  is equal to the law of  $g_1 S_\theta$  where  $g_1$  and  $S_\theta$  are assumed to be independent which is known to be  $\Gamma(1/2, \theta)$ ; here  $X \sim \Gamma(a, b)$  means that  $X$  has the Gamma density

$$p(x; a, b) = \frac{b(bx)^{a-1}}{\Gamma(a)} \exp(-bx) 1_{(0, \infty)}(x).$$

Let  $\mathbb{U}$  be the standard Brownian bridge. Let  $\gamma \sim \Gamma(1/2, \theta)$  be independent from  $\mathbb{U}$  and let

$$\tilde{\mathbb{U}}_t = \sqrt{\gamma} \mathbb{U}_{t/\gamma} \tag{1.5}$$

for  $0 \leq t \leq \gamma$ . The process  $(\tilde{\mathbb{U}}_t : 0 \leq t \leq \gamma)$  is called the randomly rescaled Brownian bridge. From the independence of  $g_{S_\theta}$  and the process defined in (1.4) we have

$$(\tilde{\mathbb{U}}_t : 0 \leq t \leq \gamma) \stackrel{d}{=} (B_t : 0 \leq t \leq g_{S_\theta}). \tag{1.6}$$

This equality in law can be exploited to derive Laplace transforms of distributions of functionals of Brownian bridge. From (1.6) it follows that

$$(\sqrt{\gamma}M^+, \sqrt{\gamma}M^-) \stackrel{d}{=} \left( \max_{0 \leq t \leq g_{S_\theta}} B_t, - \min_{0 \leq t \leq g_{S_\theta}} B_t \right). \tag{1.7}$$

Excursion theory will provide the distribution of the pair on the right in (1.7) which in turn is used to derive the Laplace transform of the cumulative distribution function of the pair  $(M^+, M^-)$ . The transform can be inverted in the form of infinite series. The method has been used widely in the literature and is well known. See [18,4,19,17] for results based on this identity in law. The contribution of this paper is a unified way to derive explicitly the distributions of functionals related to the pair  $(M^+, M^-)$ .

## 2. Brownian excursions

The paths of Brownian motion  $B$  are continuous functions, hence the complements of their zero sets are unions of disjoint open intervals. The path of Brownian motion restricted to any such open interval is called an excursion away from 0. Since Brownian motion is recurrent all open intervals will be bounded. The path can thus be broken up into an infinite string of excursions and every excursion can be identified with a function in the set of functions

$$\mathcal{U} = \{w \in C[0, \infty), w(0) = 0, \exists R > 0, w(t) \neq 0 \text{ iff } t \in (0, R)\}. \tag{2.1}$$

To describe the structure of excursions let  $(L(t) : t \geq 0)$  be the local time process at level 0 for Brownian motion normalized so that

$$L(t) \stackrel{d}{=} \max_{0 \leq s \leq t} B_s. \tag{2.2}$$

Local time is an adapted nondecreasing process which only increases on the zero set of Brownian motion and  $L(t_1) < L(t_2)$  for  $t_1 < t_2$  whenever the interval  $(t_1, t_2)$  contains a zero of  $B$ . Hence the local time during two different excursions is different and constant during each excursion. See [20] for definitions and fundamental results on local time. Let  $\tau_s = \inf\{u : L(u) > s\}$  be the right continuous inverse of the local time process  $L(t)$ . From the properties of local time we infer that every excursion of Brownian motion away from 0 is of the form

$$e_s(u) = 1_{[0 \leq u \leq \tau_s - \tau_{s-}]}(u) B_{\tau_{s-} + u} \tag{2.3}$$

for those  $s$  at which  $\tau_s$  has a jump. Let  $e$  be the point process defined on the abstract space  $(0, \infty) \times \mathcal{U}$  defined as

$$e = \{(s, e_s) : s > 0, \tau_s - \tau_{s-} > 0\}. \tag{2.4}$$

The following theorem by Itô is one of his great insights.

**Theorem 2.** *The point process  $e$  is a Poisson process on  $(0, \infty) \times \mathcal{U}$  with mean measure given by  $\lambda \times n$  where  $\lambda$  is the Lebesgue measure on  $(0, \infty)$  and  $n$  is a  $\sigma$ -finite measure on the functions space  $\mathcal{U}$  equipped with the  $\sigma$ -field generated by the coordinate maps.*

For a proof see [20, p. 457]. Note that the excursions of the process  $(B_t : 0 \leq t \leq g_{S_\theta})$  are a portion of the excursion process of Brownian motion. It will be shown that the law of this

portion can be described and used to derive the distribution of the pair of variables on the right side of (1.7).

If the points of a Poisson process on an abstract space with mean measure  $\mu$  are marked in such a way that each point receives a mark independently of other points with probability depending on the position of the point then the point processes of marked and unmarked excursions are two independent Poisson processes. If at position  $x$  a mark is assigned with probability  $f(x)$  then the marked and unmarked Poisson processes have mean measures  $f \cdot \mu$  and  $(1 - f) \cdot \mu$ . See [11, p. 55], for definitions and proof.

Marking will be applied to the Poisson process of excursions. Define the duration of an excursion  $w \in \mathcal{U}$  as

$$R(w) = \sup\{u: w(u) \neq 0\}$$

and assign marks to the points of the process of excursions with probability  $1 - e^{-\theta R(w)}$  for  $\theta > 0$ . Define  $T = \inf\{s: e_s \text{ is marked}\}$ .

**Theorem 3.** *Let  $\tilde{e}$  be the point process  $\{(s, e_s): 0 < s < T\}$ .*

- (i) *The random variable  $T$  is exponential with parameter  $\sqrt{2\theta}$ .*
- (ii) *Conditionally on  $T = t$ , the point process  $\tilde{e}$  is a Poisson process in the space  $(0, t) \times U$  with mean measure  $\lambda \times n \cdot e^{-\theta R(w)}$  where  $\lambda$  is the Lebesgue measure on  $(0, t)$  and  $n$  is Itô's excursion law for Brownian motion.*
- (iii) *Positive and negative excursions of  $\tilde{e}$  are conditionally independent Poisson processes given  $T = t$ .*

**Proof.** For the Itô measure  $n$  we have

$$n(R \in dr) = \frac{dr}{\sqrt{2\pi r^3}}.$$

See [20, p. 459]. The probability that there is no marked excursion in  $(0, t) \times \mathcal{U}$  is given by

$$\exp\left(-\int_{(0,t) \times U} (1 - e^{-\theta r}) dt n(R \in dr)\right) = e^{-t\sqrt{2\theta}}.$$

The integral above is given e.g. in [2, p. 73]. It follows that  $P(T > t) = e^{-t\sqrt{2\theta}}$  which proves (i). The assertions in (ii) and (iii) follow from independence properties of Poisson processes bearing in mind that positive and negative excursions of  $e$  are independent Poisson processes and that marking is independent of the sign of excursions.  $\square$

The law of excursions of  $(B_t: 0 \leq t \leq g_{S_\theta})$  is described in the following theorem. See also [21, p. 418].

**Theorem 4.** *The law of the point process  $((s, e_s): 0 < s < L(S_\theta))$  is described by:*

- (i)  *$L(S_\theta)$  is exponential with parameter  $\sqrt{2\theta}$ .*
- (ii) *Conditionally on  $L(S_\theta) = t$  the process  $((s, e_s): 0 < s < L(S_\theta))$  is a Poisson process on  $(0, t) \times \mathcal{U}$  with mean measure  $\lambda \times n \cdot e^{-\theta R(w)}$ .*
- (iii) *Conditionally on  $L(S_\theta) = t$  the positive and negative excursions of  $((s, e_s): 0 < s < L(S_\theta))$  are independent Poisson processes.*

**Proof.** Let  $N$  be a Poisson process with intensity  $\theta$  on  $(0, \infty)$  independent of  $B$ . If  $e_s$  is an excursion on the open interval of length  $R$  then by independence the open interval contains a point of  $N$  with probability  $1 - e^{-\theta R}$ . Declare all excursions that contain a point of  $N$  to be marked. By independence properties of  $N$  marks are assigned independently. The leftmost point of  $N$  will be an exponential random variable  $S_\theta$  independent of  $B$ . It follows that the excursion straddling  $S_\theta$  is exactly the first marked excursion of the excursion process  $e$ . Hence the excursion process of  $(B_t: 0 \leq t \leq g_{S_\theta})$  is exactly the portion of the excursion process  $e$  up to the first marked excursion. The assertions (i), (ii) and (iii) follow from [Theorem 3](#).  $\square$

Let  $M^+$ ,  $M^-$  and  $M$  be as defined in Section 1. Let  $\mathbb{U}$  be a Brownian bridge and  $\gamma$  a  $\Gamma(1/2, \theta)$  random variable independent of  $\mathbb{U}$ . Some preliminary calculations are needed to find explicitly the distribution of  $(\sqrt{\gamma}M_+, \sqrt{\gamma}M^-)$ .

The reflection principle for Brownian motion states, see [3, p. 126, formula 1.1.8], that for  $x > 0$  and  $z < y < x$

$$P\left(\max_{0 \leq t \leq 1} B_t \geq x, z < B_1 < y\right) = P(2x - y < B_1 < 2x - z). \tag{2.5}$$

Brownian bridge is Brownian motion conditioned to be 0 at time  $t = 1$  so by (2.5) for  $x > 0$

$$\begin{aligned} P(M^+ \geq x) &= \lim_{\epsilon \rightarrow 0} P\left(\max_{0 \leq t \leq 1} B_t \geq x \mid |B_1| \leq \epsilon\right) \\ &= \lim_{\epsilon \rightarrow 0} P\left(\max_{0 \leq t \leq 1} B_t \geq x, |B_1| \leq \epsilon\right) / P(|B_1| \leq \epsilon) \\ &= \lim_{\epsilon \rightarrow 0} P(2x - \epsilon \leq B_1 \leq 2x + \epsilon) / P(|B_1| \leq \epsilon) \\ &= e^{-2x^2}. \end{aligned} \tag{2.6}$$

It follows from the distribution of  $M^+$  given by (2.6) that

$$\begin{aligned} P(\sqrt{\gamma}M^+ \geq x) &= \sqrt{\theta/\pi} \int_0^\infty \exp(-2x^2/s) s^{-1/2} e^{-\theta s} ds \\ &= \exp(-2x\sqrt{2\theta}). \end{aligned} \tag{2.7}$$

The integral is given in [16, p. 41, formula 5.28].

Turning to excursions recall that  $R(w)$  stands for the length of the excursion and denote  $w^+ = \max_u w(u)$ . Define for  $x > 0$

$$m(x) = \int_{\{w^+ > x\}} e^{-\theta R(w)} n(dw).$$

**Theorem 5.** *The law of the triple*

$$\left(\max_{0 \leq t \leq g_{S_\theta}} B_t, -\min_{0 \leq t \leq g_{S_\theta}} B_t, L(S_\theta)\right)$$

is described by:

- (i)  $L(S_\theta)$  is exponential with parameter  $\sqrt{2\theta}$ .

(ii) The random variables  $\max_{0 \leq t \leq g_{S_\theta}} B_t$  and  $-\min_{0 \leq t \leq g_{S_\theta}} B_t$  are conditionally independent given  $L(S_\theta) = t$  with the same conditional distribution.

(iii)

$$P\left(\max_{0 \leq t \leq g_{S_\theta}} B_t \leq x \mid L(S_\theta) = t\right) = e^{-tm(x)}.$$

(iv)

$$m(x) = \frac{\sqrt{2\theta} e^{-2x\sqrt{2\theta}}}{1 - e^{-2x\sqrt{2\theta}}}. \tag{2.8}$$

**Proof.** (i) is proved in Theorem 4. The point processes of positive and negative excursions of  $((s, e_s): 0 < s < L(S_\theta))$  are conditionally independent given  $L(S_\theta) = t$  by Theorem 4. Since  $\max_{0 \leq t \leq g_{S_\theta}} B_t$  is a function of positive and  $-\min_{0 \leq t \leq g_{S_\theta}} B_t$  a function of negative excursions conditional independence follows. Equality of conditional distributions follows by symmetry. The process  $((s, e_s^+): 0 < s < L(S_\theta))$  is a measurable map of the process  $((s, e_s): 0 < s < L(S_\theta))$  hence conditionally on  $L(S_\theta) = t$  a Poisson process on  $(0, t) \times (0, \infty)$ ; see [11]. Conditionally on  $L(S_\theta) = t$  we have that  $\max_{0 \leq t \leq g_{S_\theta}} B_t \leq x$  if there is no point of  $((s, e_s^+): 0 < s < L(S_\theta))$  in the set  $(0, t) \times (x, \infty)$ . The measure of this set is  $tm(x)$  by Theorem 4(ii). The assertion (iii) follows. By unconditioning

$$\begin{aligned} P(\max_{0 \leq t \leq g_{S_\theta}} B_t > x) &= \sqrt{2\theta} \int_0^\infty e^{-\sqrt{2\theta}t} (1 - e^{-tm(x)}) dt \\ &= \frac{m(x)}{\sqrt{2\theta} + m(x)}. \end{aligned} \tag{2.9}$$

Comparing (2.7) and (2.9) we obtain that

$$\frac{m(x)}{\sqrt{2\theta} + m(x)} = \exp\left(-2x\sqrt{2\theta}\right). \tag{2.10}$$

(iv) follows by solving for  $m(x)$ .  $\square$

The law of the triple is in accordance with formula (53) in [19].

### 3. Examples of calculations

#### 3.1. Distributions of $M$ and $m$

Let  $m$  and  $M$  be defined as in (1.2). By (1.7) the random variables  $\max_{0 \leq t \leq g_{S_\theta}} B_t$  and  $\sqrt{\gamma}M^+$  have the same distribution. We compute, by conditioning on  $L(S_\theta)$  and using Theorem 5,

$$\begin{aligned} P(\sqrt{\gamma}M \leq x) &= P(\sqrt{\gamma}M^+ \leq x, \sqrt{\gamma}M^- \leq x) \\ &= P\left(\max_{0 \leq t \leq g_{S_\theta}} B_t \leq x, -\min_{0 \leq t \leq g_{S_\theta}} B_t \leq x\right) \\ &= E\left(P\left(\max_{0 \leq t \leq g_{S_\theta}} B_t \leq x, -\min_{0 \leq t \leq g_{S_\theta}} B_t \leq x \mid L(S_\theta)\right)\right) \\ &= E(\exp(-2L(S_\theta)m(x))) \end{aligned}$$

$$\begin{aligned}
 &= \int_0^\infty \sqrt{2\theta} \exp(-\sqrt{2\theta}v) \exp(-2vm(x)) dx \\
 &= \frac{\sqrt{2\theta}}{\sqrt{2\theta} + 2m(x)} = \tanh(x\sqrt{2\theta})
 \end{aligned} \tag{3.1}$$

where we used (2.8) in the last step. Thus

$$P(\sqrt{\gamma}M \leq x) = \frac{\sqrt{2\theta}}{\sqrt{2\theta} + 2m(x)} = \tanh(x\sqrt{2\theta}). \tag{3.2}$$

Let  $F_M$  be the cumulative distribution function of  $M$ . Writing out (3.2), taking into account the independence of  $M$  and  $\gamma$  and dividing both sides by  $\sqrt{\theta}$  we get

$$\frac{1}{\sqrt{\pi}} \int_0^\infty F_M(x/\sqrt{s}) s^{-1/2} e^{-\theta s} ds = \frac{\tanh(x\sqrt{2\theta})}{\sqrt{\theta}}. \tag{3.3}$$

Oberhettinger and Badii [16, p. 294, formula 8.51], give the inverse of the Laplace transform on the right of (3.3) as

$$\begin{aligned}
 \frac{1}{\sqrt{\pi}} F_M(x/\sqrt{s}) s^{-1/2} &= \frac{1}{\sqrt{2x}} \theta_2(0|s/(2x^2)) \\
 &= \frac{1}{\sqrt{2x}} \frac{\sqrt{2x}}{\sqrt{\pi s}} \sum_{k=-\infty}^\infty (-1)^k \exp(-2k^2x^2/s).
 \end{aligned}$$

This yields

$$F_M(z) = \sum_{k=-\infty}^\infty (-1)^k \exp(-2k^2z^2),$$

or

$$1 - F_M(z) = 2 \sum_{k=1}^\infty (-1)^{k+1} \exp(-2k^2z^2),$$

which is the formula for the distribution of the Kolmogorov–Smirnov test statistic.

Turning to  $m = \min\{M^+, M^-\}$  observe that by Theorem 5

$$P(\sqrt{\gamma}m > x) = \sqrt{2\theta} \int_0^\infty e^{-\sqrt{2\theta}t} (1 - e^{-tm(x)})^2 dt. \tag{3.4}$$

Integration yields

$$P(\sqrt{\gamma}m > x) = 1 - \frac{2\sqrt{2\theta}}{\sqrt{2\theta} + m(x)} + \frac{\sqrt{2\theta}}{\sqrt{2\theta} + 2m(x)}. \tag{3.5}$$

Denote by  $F_m$  the distribution function of  $m$ . From (3.5), independence of  $m$  and  $\gamma$  and dividing both sides by  $\sqrt{\theta}$  we get

$$\frac{1}{\sqrt{\pi}} \int_0^\infty (1 - F_m(x/\sqrt{s})) s^{-1/2} e^{-\theta s} ds = \frac{\tanh(x\sqrt{2\theta})}{\sqrt{\theta}} + \frac{2e^{-2x\sqrt{2\theta}}}{\sqrt{\theta}} - \frac{1}{\sqrt{\theta}}.$$



The first term has been inverted above, the second is given by Oberhettinger and Badii [16, p. 258, formula 5.87], and the third is elementary. Substituting  $z$  for  $x/\sqrt{s}$  one gets

$$\begin{aligned}
 1 - F_m(z) &= \sum_{k=-\infty}^{\infty} (-1)^k \exp(-2k^2 z^2) + 2e^{-2z^2} - 1 \\
 &= 2 \sum_{k=2}^{\infty} (-1)^k \exp(-2k^2 z^2).
 \end{aligned}
 \tag{3.6}$$

### 3.2. Joint distributions, sums, differences, quotients

In this section the distributions of various functions of the pair  $(M^+, M^-)$  will be derived. Let  $K \equiv M^+ + M^-$ , the Kuiper (or range) statistic,  $L \equiv M^+ - M^-$ , the difference statistic, and let  $Q \equiv M^+/M^-$ , the ratio statistic.

**Theorem 6.** (i) *The joint distribution of  $(M^+, M^-)$  is given for  $z, w > 0$  by*

$$\begin{aligned}
 P(M^+ \leq z, M^- \leq w) &= \sum_{k=-\infty}^{\infty} \exp(-2k^2(z+w)^2) \\
 &\quad - \sum_{k=-\infty}^{\infty} \exp(-2(k(z+w) + z)^2).
 \end{aligned}
 \tag{3.7}$$

(ii) For  $x > 0$ ,

$$F_K(x) = P(M^+ + M^- \leq x) = \sum_{k=-\infty}^{\infty} (1 - 4k^2 x^2) e^{-2k^2 x^2}.
 \tag{3.8}$$

(iii) For  $x > 0$ ,

$$1 - F_L(x) = P(M^+ - M^- \geq x) = \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} \cdot e^{-2k^2 x^2}.
 \tag{3.9}$$

(iv) For  $x > 0$ ,

$$F_Q(x) = P(Q \leq x) = \frac{1}{z+1} \left( 1 - \frac{\pi z \cot\left(\frac{\pi z}{z+1}\right)}{z+1} \right).
 \tag{3.10}$$

**Remark 1.** The formula for the joint distribution in (i) is in agreement with Shorack and Wellner [23, formula (2.2.22), p. 39].

**Remark 2.** The result (ii) is in agreement with Kuiper [14] and with Dudley [6, Proposition 22.10, p. 22.6]. Vervaat [24] gives a construction of standard Brownian excursion from a Brownian bridge. Let  $\mathbb{U}$  be a Brownian bridge  $[0, 1]$  and let  $\sigma$  be the time when  $\mathbb{U}$  attains its minimum on  $[0, 1]$  ( $\sigma$  is a.s. unique). Then the process  $(e(t): 0 \leq t \leq 1)$  defined by

$$e(t) = \mathbb{U}_{\sigma+t \pmod{1}} - \mathbb{U}_{\sigma}(t)$$

is a standard Brownian excursion. It is a simple consequence of this transformation that the Kuiper statistic has the distribution of the maximum of the standard Brownian excursion and

(3.8) is another derivation of the distribution of this maximum. Further results for range statistics are given by Feller [8] and Salminen and Vallois [22].

**Remark 3.** Note that the distribution of  $M^+ - M^-$  is symmetric about 0. For a different approach for Brownian motion instead of Brownian bridge see [13].

**Remark 4.** The result in (iv) is in accordance with the distribution for the ratio  $\tilde{Q} = M^+ / (M^+ + M^-)$  given in [5]. The derivation of the distribution of  $\tilde{Q}$  based on rescaling arguments is given in [19] and can be derived from the above result for  $F_Q$ .

**Proof.** By unconditioning in Theorem 5

$$\begin{aligned}
 P(\sqrt{\gamma}M^+ \leq x, \sqrt{\gamma}M^- \leq y) &= \frac{\sqrt{2\theta}}{\sqrt{2\theta} + m(x) + m(y)} \\
 &= \frac{(1 - e^{-2x\sqrt{2\theta}})(1 - e^{-2y\sqrt{2\theta}})}{1 - e^{-2x\sqrt{2\theta}}e^{-2y\sqrt{2\theta}}} \\
 &= \frac{(e^{x\sqrt{2\theta}} - e^{-x\sqrt{2\theta}})(e^{y\sqrt{2\theta}} - e^{-y\sqrt{2\theta}})}{e^{(x+y)\sqrt{2\theta}} - e^{-(x+y)\sqrt{2\theta}}} \\
 &= 2 \frac{\sinh(x\sqrt{2\theta}) \sinh(y\sqrt{2\theta})}{\sinh((x+y)\sqrt{2\theta})} \\
 &= \frac{\cosh((x+y)\sqrt{2\theta}) - \cosh((x-y)\sqrt{2\theta})}{\sinh((x+y)\sqrt{2\theta})} \\
 &= \coth((x+y)\sqrt{2\theta}) - \frac{\cosh((x-y)\sqrt{2\theta})}{\sinh((x+y)\sqrt{2\theta})}. \tag{3.11}
 \end{aligned}$$

Let  $F(z, w)$  be the joint distribution function of the pair  $(M^+, M^-)$ . By independence of  $\gamma$  and  $(M^+, M^-)$

$$P(\sqrt{\gamma}M^+ \leq x, \sqrt{\gamma}M^- \leq y) = \sqrt{\theta/\pi} \int_0^\infty F(x/\sqrt{s}, y/\sqrt{s}) s^{-1/2} e^{-\theta s} ds. \tag{3.12}$$

The right side is given in (3.11). Oberhettinger and Badii [16, p. 294, formula 8.52], give the inverse of the first term on the right in (3.11)

$$\frac{1}{\sqrt{2}(x+y)} \theta_3 \left( 0 \left| \frac{s}{2(x+y)^2} \right. \right) = \frac{1}{\sqrt{\pi s}} \sum_{k=-\infty}^\infty e^{-2k^2(x+y)^2/s}.$$

The inverse of the second term of the transform can be obtained from Oberhettinger and Badii [16, p. 294, formula 8.60]: we find that the inverse is

$$\begin{aligned}
 &\frac{1}{\sqrt{2}(x+y)} \theta_4 \left( \frac{(x-y)/2}{x+y} \left| \frac{s}{2(x+y)^2} \right. \right) \\
 &= \frac{1}{\sqrt{\pi s}} \sum_{k=-\infty}^\infty \exp \left( -2(x+y)^2 \left[ \frac{(x-y)/2}{(x+y)} + k + \frac{1}{2} \right]^2 /s \right) \\
 &= \frac{1}{\sqrt{\pi s}} \sum_{k=-\infty}^\infty \exp(-2[k(x+y) + x]^2/s).
 \end{aligned}$$

Combining these yields

$$P(M^+ \leq z, M^- \leq w) = \sum_{k=-\infty}^{\infty} \exp(-2k^2(z+w)^2) - \sum_{k=-\infty}^{\infty} \exp(-2(k(z+w) + z)^2).$$

We now consider the Kuiper statistic  $K = M^+ + M^-$ . It seems cumbersome to proceed from the joint distribution of  $M^+$  and  $M^-$  so we use directly the distribution of  $(\sqrt{\gamma}M^+, \sqrt{\gamma}M^-)$ . Denote  $U = \sqrt{\gamma}M^+$  and  $V = \sqrt{\gamma}M^-$ . The joint cumulative distribution function of  $U$  and  $V$  is given in (3.11) as

$$G(u, v) = 2 \frac{\sinh(u\sqrt{2\theta}) \sinh(v\sqrt{2\theta})}{\sinh((u+v)\sqrt{2\theta})}.$$

The cumulative distribution function of  $U + V$  is given by

$$P(U + V \leq z) = \int_0^z G_u(u, z-u) du$$

where  $G_u$  is the partial derivative of  $G$  with respect to  $u$ . A calculation yields

$$G_u(u, z-u) = \frac{2\sqrt{2\theta} \sinh^2((z-u)\sqrt{2\theta})}{\sinh^2(z\sqrt{2\theta})}$$

and integration gives

$$P(U + V \leq z) = \coth(z\sqrt{2\theta}) - \frac{z\sqrt{2\theta}}{\sinh^2(z\sqrt{2\theta})}.$$

Using the fact that  $\gamma$  and  $M^+ + M^-$  are independent we obtain the Laplace transform of the cumulative distribution function  $F_K$  of the Kuiper statistic as

$$\frac{\sqrt{\theta}}{\sqrt{\pi}} \int_0^\infty F_K(z/\sqrt{s}) s^{-1/2} e^{-\theta s} ds = \coth(z\sqrt{2\theta}) - \frac{z\sqrt{2\theta}}{\sinh^2(z\sqrt{2\theta})}.$$

After dividing by  $\sqrt{\theta}$  it remains to invert the two terms on the right and substitute for  $z/\sqrt{s}$ . The first term has been inverted above when deriving the joint distribution of  $M^+$  and  $M^-$ . We get

$$\frac{1}{\sqrt{2z}} \theta_3 \left( 0 \left| \frac{s}{2z^2} \right. \right) = \frac{1}{\sqrt{\pi s}} \sum_{k=-\infty}^{\infty} e^{-2k^2 z^2 / s}.$$

To invert the second term rewrite it as

$$\frac{\sqrt{2z}}{\sinh^2(z\sqrt{2\theta})} = \frac{4\sqrt{2z} e^{-2z\sqrt{2\theta}}}{(1 - e^{-2z\sqrt{2\theta}})^2} = 4\sqrt{2z} \sum_{k=1}^{\infty} k e^{-2kz\sqrt{2\theta}}$$

for  $z > 0$  and  $\theta > 0$ . The inverse Laplace transforms of the terms in the sum are known, see [16, p. 258, formula 5.85]. Taking the derivative with respect to  $x$  on both sides of (2.7) we get

$$\int_0^\infty \frac{a}{\sqrt{2\pi s^3}} \exp(-a^2/2s) e^{-\theta s} ds = e^{-a\sqrt{2\theta}}. \tag{3.13}$$

Since all the terms are nonnegative functions the order of summation and integration can be changed. Hence for  $z > 0$  the inverse Laplace transform of the second term is

$$4\sqrt{2z} \sum_{k=1}^{\infty} \frac{2k^2 z}{\sqrt{2\pi s^3}} \exp(-2k^2 z^2 / s) = \frac{1}{\sqrt{\pi s}} \sum_{k=-\infty}^{\infty} \frac{4k^2 z^2}{s} \exp(-2k^2 z^2 / s).$$

Substitute  $x = z/\sqrt{s}$  to get

$$F_K(x) = \sum_{k=-\infty}^{\infty} e^{-2k^2x^2} - \sum_{k=-\infty}^{\infty} 4k^2x^2e^{-2k^2x^2} = \sum_{k=-\infty}^{\infty} (1 - 4k^2x^2)e^{-2k^2x^2}. \tag{3.14}$$

For the difference  $U - V$  a computation yields for  $z > 0$

$$P(U - V \geq z) = \int_z^{\infty} G_u(u, u - z) du \tag{3.15}$$

provided  $P(U > 0) = 1$  and  $P(V > 0) = 1$  which is the case for the variables in question. Using the joint cumulative distribution function yields

$$G_u(u, u - z) = 2\sqrt{2\theta} \frac{\sinh^2(\sqrt{2\theta}(u - z))}{\sinh^2(\sqrt{2\theta}(2u - z))}.$$

From (3.15) it follows

$$\begin{aligned} P(U - V \geq z) &= 2\sqrt{2\theta} \int_z^{\infty} \frac{\sinh^2(\sqrt{2\theta}(u - z))}{\sinh^2(\sqrt{2\theta}(2u - z))} du \\ &= 2\sqrt{2\theta} \int_0^{\infty} \frac{\sinh^2(\sqrt{2\theta}u)}{\sinh^2(\sqrt{2\theta}(2u + z))} du. \\ &= \frac{1}{2} - \operatorname{arctanh}(e^{-\sqrt{2\theta}z}) \sinh(\sqrt{2\theta}z). \end{aligned}$$

The integral in the last line is elementary and is computed by Mathematica. Since  $\gamma$  and  $(M^+, M^-)$  are independent we have for  $z > 0$

$$\begin{aligned} P(U - V \geq z) &= \frac{\sqrt{\theta}}{\sqrt{\pi}} \int_0^{\infty} P\left(M^+ - M^- \geq \frac{z}{\sqrt{s}}\right) \frac{e^{-\theta s}}{\sqrt{s}} ds \\ &= \frac{1}{2} - \operatorname{arctanh}\left(e^{-\sqrt{2\theta}z}\right) \sinh\left(\sqrt{2\theta}z\right). \end{aligned} \tag{3.16}$$

This gives the Laplace transform of  $P(M^+ - M^- \geq z/\sqrt{s})/\sqrt{s}$  as a function of  $s$  for fixed  $z$ .

To invert this Laplace transform we use the known series expansion for the hyperbolic arc-tangent to get for  $z > 0$

$$\frac{1}{2} - \operatorname{arctanh}\left(e^{-\sqrt{2\theta}z}\right) \sinh\left(\sqrt{2\theta}z\right) = \sum_{k=1}^{\infty} \frac{e^{-2kz\sqrt{2\theta}}}{4k^2 - 1}. \tag{3.17}$$

Finally

$$\int_0^{\infty} P\left(M^+ - M^- \geq \frac{z}{\sqrt{s}}\right) \frac{e^{-\theta s}}{\sqrt{s}} ds = \sum_{k=1}^{\infty} \frac{(\pi/\theta)^{1/2} e^{-2kz\sqrt{2\theta}}}{4k^2 - 1}.$$

All the terms in the sum are Laplace transforms and Oberhettinger and Badii [16, p. 258, formula 5.87], give the inverses. Since all the terms are positive the series can be inverted termwise and we get

$$\frac{1}{\sqrt{s}} P\left(M^+ - M^- \geq \frac{z}{\sqrt{s}}\right) = \frac{1}{\sqrt{s}} \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} \cdot e^{-2k^2z^2/s} \tag{3.18}$$

or

$$P(M^+ - M^- \geq z) = \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} \cdot e^{-2k^2 z^2}.$$

We now turn to the quotient  $Q = M^+ / M^-$ . We can multiply the numerator and denominator by  $\sqrt{\gamma}$  and choose  $\theta = 1/2$ . Conditionally on  $L(S_\theta) = t$  an elementary calculation gives the conditional distribution function of  $Q$  as

$$F_{Q|L(S_\theta)=t}(z) = - \int_0^\infty t m'(x) e^{-tm(x)} e^{-tm(xz)} dx.$$

Unconditioning and changing the order of integration gives

$$F_Q(z) = - \int_0^\infty \frac{m'(x) dx}{(1 + m(x) + m(xz))^2}. \tag{3.19}$$

Substituting (2.8) and observing that

$$m'(x) = - \frac{1}{2 \sinh^2 x}$$

gives

$$\begin{aligned} F_Q(z) &= \frac{1}{2} \int_0^\infty \frac{(1 - e^{-2x})^2 (1 - e^{-2xz})^2 dx}{\sinh^2 x (1 - e^{-2x(z+1)})^2} \\ &= 2 \int_0^\infty \frac{\sinh^2 zx dx}{\sinh^2 ((z+1)x)} \\ &= \frac{1}{z+1} \left( 1 - \frac{\pi z \cot\left(\frac{\pi z}{z+1}\right)}{z+1} \right) \end{aligned}$$

where the last integral is given in [10, formula 3.511.9].  $\square$

### 3.3. Covariance and correlation

The correlation between  $M^+$  and  $M^-$  may be a quantity of interest. Let  $U = \max_{0 \leq t \leq g_{S_\theta}} B_t$  and  $V = - \min_{0 \leq t \leq g_{S_\theta}} B_t$ . The covariance of  $U$  and  $V$  will be computed first. By symmetry

$$E(U|L(S_\theta)) = E(V|L(S_\theta)).$$

We have

$$\text{cov}(U, V) = E(\text{cov}(U, V|L(S_\theta))) + \text{cov}(E(U|L(S_\theta)), E(V|L(S_\theta))). \tag{3.20}$$

Denote  $E(U|L(S_\theta)) = \psi(L(S_\theta))$ . By Theorem 5(iii), given  $L(S_\theta)$  the conditional covariance of  $U$  and  $V$  is 0. It follows

$$\text{cov}(U, V) = \text{var}(\psi(L(S_\theta))).$$

The conditional expectation is computed from Theorem 5(iii), as

$$\psi(t) = \int_0^\infty (1 - e^{-tm(x)}) dx. \tag{3.21}$$

We can choose  $2\theta = 1$  so that the local time  $L(S_\theta)$  is exponential with parameter 1 and compute

$$\begin{aligned}
 E(U) &= \int_0^\infty e^{-t} \psi(t) dt \\
 &= \int_0^\infty e^{-t} dt \int_0^\infty (1 - e^{-tm(x)}) dx \\
 &= \int_0^\infty \frac{m(x)}{1 + m(x)} dx \\
 &= \int_0^\infty e^{-2x} dx \\
 &= 1/2
 \end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
 E(\psi^2(L(S_{1/2}))) &= \int_0^\infty e^{-t} dt \int_0^\infty (1 - e^{-tm(x)}) dx \int_0^\infty (1 - e^{-tm(y)}) dy \\
 &= \int_0^\infty dx \int_0^\infty dy \left[ 1 - \frac{1}{1 + m(x)} - \frac{1}{1 + m(y)} + \frac{1}{1 + m(x) + m(y)} \right] \\
 &= \int_0^\infty dx \int_0^\infty dy \frac{e^{-2(x+y)}(2 - e^{-2x} - e^{-2y})}{1 - e^{-2(x+y)}} \\
 &= \int_0^\infty dt \frac{e^{-2t}}{1 - e^{-2t}} \int_0^t (2 - e^{-2(t-v)} - e^{-2v}) dv \\
 &= \int_0^\infty \frac{e^{-2t}}{1 - e^{-2t}} (2t - (1 - e^{-2t})) dt \\
 &= 2 \int_0^\infty \sum_{k=1}^\infty t e^{-2kt} dt - \frac{1}{2} \\
 &= 2 \sum_{k=1}^\infty \frac{1}{4k^2} - \frac{1}{2} \\
 &= \frac{\pi^2}{12} - \frac{1}{2}.
 \end{aligned} \tag{3.23}$$

The second line follows from Fubini's theorem and the third by changing variables to  $x + y = t$ ,  $y = v$ . It follows

$$\text{cov}(U, V) = \text{var}(\psi(L(S_\theta))) = \frac{\pi^2}{12} - \frac{3}{4}.$$

From (2.6) one derives  $E(M^+) = E(M^-) = \sqrt{2\pi}/4$  and  $\text{var}(M^+) = \text{var}(M^-) = 1/2 - \pi/8$ . By (1.7) and the independence of  $\gamma$  and  $(M^+, M^-)$  we have

$$\text{cov}(U, V) = E(\gamma) E(M^+ M^-) - E^2(\sqrt{\gamma}) E(M^+) E(M^-).$$

It follows

$$E(M^+ M^-) = \frac{\pi^2}{12} - \frac{1}{2}$$

and

$$\text{cov}(M^+, M^-) = \frac{\pi^2}{12} - \frac{1}{2} - \frac{\pi}{8} \quad (3.24)$$

yielding

$$\text{corr}(M^+, M^-) = \frac{\frac{\pi^2}{12} - \frac{1}{2} - \frac{\pi}{8}}{\frac{1}{2} - \frac{\pi}{8}} = -0.654534. \quad (3.25)$$

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## References

- [1] M. Barlow, J. Pitman, M. Yor, Une extension multidimensionnelle de la loi de l'arc sinus, in: Séminaire de Probabilités, XXIII, in: Lecture Notes in Math., vol. 1372, Springer, Berlin, 1989, pp. 294–314.
- [2] J. Bertoin, Lévy Processes, in: Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996.
- [3] A.N. Borodin, P. Salminen, Handbook of Brownian Motion—Facts and Formulae, first ed., in: Probability and its Applications, Birkhäuser Verlag, 1996.
- [4] P. Carmona, F. Petit, J. Pitman, M. Yor, On the laws of homogeneous functionals of the Brownian bridge, *Studia Sci. Math. Hungar.* 35 (1999) 445–455.
- [5] E. Csáki, On some distributions concerning maximum and minimum of a Wiener process, in: Analytic Function Methods in Probability Theory (Proc. Colloq. Methods of Complex Anal. in the Theory of Probab. and Statist., Kossuth L. Univ. Debrecen, Debrecen, 1977), in: Colloq. Math. Soc. János Bolyai, vol. 21, North-Holland, Amsterdam, 1979, pp. 43–52.
- [6] R.M. Dudley, Probabilities and Metrics: Convergence of Laws on Metric Spaces, with A View to Statistical Testing, in: Lecture Notes Series, vol. 45, Matematisk Institut, Aarhus Universitet, Aarhus, 1976.
- [7] E.B. Dynkin, Some limit theorems for sums of independent random variables with infinite mathematical expectations, in: Select. Transl. Math. Statist. and Probability, Vol. 1, Inst. Math. Statist. and Amer. Math. Soc., Providence, RI, 1961, pp. 171–189.
- [8] W. Feller, The asymptotic distribution of the range of sums of independent random variables, *Ann. Math. Statist.* 22 (1951) 427–432.
- [9] Z. Govindarajulu, J.H. Klotz, A note on the asymptotic distribution of the one-sample Kolmogorov–Smirnov statistic, *Amer. Statist.* 27 (1973) 164–165.
- [10] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, fifth ed., Academic Press Inc., Boston, MA, 1994, Translation edited and with a preface by Alan Jeffrey.
- [11] J.F.C. Kingman, Poisson Processes, in: Oxford Science Publications, vol. 3, The Clarendon Press, Oxford University Press, New York, 1993.
- [12] A.N. Kolmogorov, Sulla determinazione empirica di una legge di distribuzione, *Giorn. Ist. Ital. Attuari* 4 (1933) 83–91.
- [13] M.R. Kosorok, C.-Y. Lin, The versatility of function-indexed weight-rank statistics, *J. Amer. Statist. Assoc.* 94 (1999) 320–332.
- [14] N.H. Kuiper, Tests concerning random points on a circle, *Nederl. Akad. Wetensch. Proc. Ser. A* 63 = *Indag. Math.* 22 (1960) 38–47.
- [15] P. Lévy, Processus Stochastiques et Mouvement Brownien. Suivi d'une Note de M. Loève, Gauthier-Villars, Paris, 1948.
- [16] F. Oberhettinger, L. Badii, Tables of Laplace Transforms, Springer-Verlag, New York, 1973.
- [17] M. Perman, J.A. Wellner, On the distribution of Brownian areas, *Ann. Appl. Probab.* 6 (1996) 1091–1111.
- [18] J. Pitman, M. Yor, Random Brownian scaling identities and splicing of Bessel processes, *Ann. Probab.* 26 (1998) 1683–1702.
- [19] J. Pitman, M. Yor, Path decompositions of a Brownian bridge related to the ratio of its maximum and amplitude, *Studia Sci. Math. Hungar.* 35 (1999) 457–474.

- [20] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, third ed., in: Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 293, Springer-Verlag, Berlin, 1999.
- [21] L.C.G. Rogers, D. Williams, Diffusions, Markov Processes, and Martingales: Itô Calculus. Vol. 2, in: Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1987.
- [22] P. Salminen, P. Vallois, On maximum increase and decrease of Brownian motion, *Ann. Inst. H. Poincaré* 43 (2007) 655–676.
- [23] G.R. Shorack, J.A. Wellner, Empirical Processes with Applications to Statistics, in: Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1986.
- [24] W. Vervaat, A relation between Brownian bridge and Brownian excursion, *Ann. Probab.* 7 (1979) 143–149.