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DISTRIBUTIONS RELATED TO LINEAR BOUNDS FOR THE EMPIRICAL DISTRIBUTION FUNCTION

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$X_1, \cdots, X_n$ are i.i.d. Uniform (0, 1) rv's with empirical df Γ_n and order statistics $0 < U_1 < \cdots < U_n < 1$. Define random variables U_*, i_* (for $n \geqq 2$) by

$$\max_{1 \leqq i \leqq n-1} \frac{U_{i+1}}{i} = \frac{U_{i_*+1}}{i_*}, \quad U_* = U_{i_*+1};$$

$i_* + 1$ is the (random) index of the order statistic at which the maximum is achieved and U_* is the value of that order statistic. The distributions of (U_*, i_*) and of U_* and i_* are found for all $n \geqq 2$, extending and complementing earlier results due to Birnbaum and Pyke, Chang, and Dempster. The limiting distributions are found and related to the corresponding sums of exponential rv's by a Poisson type invariance result for the empirical df Γ_n and its inverse Γ_n^{-1}.

1. Introduction. Let $X_1, \cdots, X_n$ be i.i.d. Uniform (0, 1) rv's with empirical df Γ_n; denote the order statistics of the sample by $0 \equiv U_0 < U_1 < \cdots < U_n < U_{n+1} \equiv 1$. Define three pairs of random variables (U^*, i^*), (U_*, i_*), (U_{**}, i_{**}) by

$$\min_{1 \leqq i \leqq n} \frac{U_i}{i} = \frac{U_{i^*}}{i^*}, \qquad U^* = U_{i^*},$$

$$\max_{1 \leqq i \leqq n-1} \frac{U_{i+1}}{i} = \frac{U_{i_*+1}}{i_*}, \qquad U_* = U_{i_*+1},$$

$$\max_{1 \leqq i \leqq n} \frac{U_i}{i} = \frac{U_{i_{**}}}{i_{**}}, \qquad U_{**} = U_{i_{**}}.$$

All three pairs are a.s. defined uniquely. These pairs of random variables are of interest for a variety of reasons, but especially because of their connections with upper and lower linear bounds for Γ_n: for $0 < \beta < 1$

$$A_n(\beta) \equiv \{\Gamma_n(t) \leqq t/\beta \text{ for all } 0 < t < 1\} = \left\{\min_{1 \leqq i \leqq n} \frac{nU_i}{i} \geqq \beta\right\};$$

and, for $0 < \beta < \infty$,

$$B_n(\beta) \equiv \{\Gamma_n(t) \geqq \beta t \text{ for all } U_1 \leqq t \leqq (1/\beta) \wedge 1\} = \left\{\max_{1 \leqq i \leqq n-1} \frac{nU_{i+1}}{i} \leqq 1/\beta\right\} \quad \text{a.s.}$$

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It is of some importance that for each $\varepsilon > 0$ one may choose $\beta = \beta_\varepsilon$ so that $P(A_n(\beta) \cap B_n(\beta)) > 1 - \varepsilon$; see Shorack (1972) for another statement of this fact (Lemma A3) and applications. Although the pair (U_{**}, i_{**}) is not connected with a linear bound for Γ_n, it is a natural counterpart to the pair (U^*, i^*), and we include it here for completeness.

Our main purpose is to find the distribution functions of (U_*, i_*) and (U_{**}, i_{**}) as well as the related marginal distributions. Dempster (1959) has found the corresponding distributions of (U^*, i^*). We also obtain the limiting distributions for all three pairs of random variables, and relate the limits to the distributions of the corresponding functionals of a standard Poisson process via an appropriate functional limit theorem. Our method is essentially that of Dempster (1959); we also use results of Chang (1955) and Eicker (1970). For related material we refer the reader to the excellent review by Durbin (1973).

2. The main results.

For $n \geqq 2$ and $u \in [0, 1]$ set

$$G^*(u, j) = P(U^* \leqq u, i^* = j), \qquad G_*(u, j) = P(U_* \leqq u, i_* = j),$$
$$H^*(u) = P(U^* \leqq u), \qquad H_*(u) = P(U_* \leqq u),$$
$$p^*(j) = P(i^* = j), \qquad p_*(j) = P(i_* = j),$$
$$G_{**}(u, j) = P(U_{**} \leqq u, i_{**} = j),$$
$$H_{**}(u) = P(U_{**} \leqq u),$$
$$p_{**}(j) = P(i_{**} = j),$$

where $j \in \{1, \cdots, n\}$ for $G^*(\cdot, j)$, $p^*(j)$, $G_{**}(\cdot, j)$, $p_{**}(j)$, and $j \in \{1, \cdots, n-1\}$ for $G_*(\cdot, j)$, $p_*(j)$; we have suppressed the sample size n in this notation. We also refer to G^*, G_*, and G_{**} as distribution functions even though they are clearly subdistribution functions; the corresponding distribution functions are easily obtained by summing on j. As a point of reference and for purposes of comparison we state the result of Dempster (1959) for (U^*, i^*) as Theorem 1. Theorems 2 and 3 give the corresponding results for (U_*, i_*) and (U_{**}, i_{**}). Theorems 4, 5, and 6 give the asymptotic distributions of (nU^*, i^*) (nU_*, i_*), and (nU_{**}, i_{**}) respectively. Theorems 7 and 8 relate these limiting distributions to those of the corresponding functionals of a standard Poisson process, N, and its inverse (or sum) process, N^{-1}. Proofs of the theorems are given in Section 3.

THEOREM 1 (Dempster). *For $n \geqq 1$ the distributions G^*, H^*, and p^* are given by*

$$G^*(u, j) = \frac{1}{j} \binom{n}{j} \left(u \wedge \frac{j}{n}\right)^j \left(1 - \left(u \wedge \frac{j}{n}\right)\right)^{n-j},$$
$$H^*(u) = \sum_{j=1}^{n} \frac{1}{j} \binom{n}{j} \left(u \wedge \frac{j}{n}\right)^j \left(1 - \left(u \wedge \frac{j}{n}\right)\right)^{n-j},$$

and,

$$p^*(j) = \frac{1}{j} \binom{n}{j} \left(\frac{j}{n}\right)^j \left(1 - \frac{j}{n}\right)^{n-j},$$

$0 \leqq u \leqq 1, j = 1, \cdots, n.$

THEOREM 2. *For $n \geqq 2$ the distributions G_*, H_*, and p_* are given by*

$$G_*(u, j) = \frac{(j-1)^{j-1}}{j^j} \sum_{l=j+1}^{n} \binom{n}{l} u^l (1-u)^{n-l} - \sum_{k=j+1}^{n-1} \binom{k-1}{j-1} \frac{(j-1)^{j-1}(k-j)^{k-j-1}}{k^k} \sum_{l=k}^{n} \binom{n}{l} \left(\frac{k}{j} u \wedge 1\right)^l \times \left(1 - \frac{k}{j} u \wedge 1\right)^{n-l},$$

$$H_*(u) = \sum_{j=1}^{n-1} G_*(u, j),$$

and,

$$p_*(j) = \frac{(j-1)^{j-1}}{j^j} - \sum_{k=j+1}^{n-1} \binom{k-1}{j-1} (j-1)^{j-1}(k-j)^{k-j-1}/k^k,$$

$0 \leqq u \leqq 1$, $j = 1, \cdots, n-1$.

(Here and in the following we use the convention that a sum from an integer l to a strictly smaller integer k is zero: for example, $G_*(u, n-1) = ((j-1)^{j-1}/j^j)u^n$; the second sum, from $k = n$ to $k = n-1$, is zero.)

THEOREM 3. *For $n \geqq 1$ the distributions G_{**}, H_{**}, and p_{**} are given by*

$$G_{**}(u, j) = \frac{1}{j} \sum_{l=j}^{n} \binom{n}{l} u^l (1-u)^{n-l} - \sum_{k=j+1}^{n} \binom{k}{j} \frac{1}{k-1} \frac{j^j (k-j)^{k-j-1}}{k^k} \sum_{l=k}^{n} \binom{n}{l} \left(\frac{k}{j} u \wedge 1\right)^l \times \left(1 - \frac{k}{j} u \wedge 1\right)^{n-l},$$

$$H_{**}(u) = \sum_{j=1}^{n} G_{**}(u, j),$$

and,

$$p_{**}(j) = \frac{1}{j} - \sum_{k=j+1}^{n} \binom{k}{j} \frac{1}{k-1} \frac{j^j (k-j)^{k-j-1}}{k^k},$$

$0 \leqq u \leqq 1$, $j = 1, \cdots, n$.

THEOREM 4. *For $u \geqq 0$ and $j \geqq 1$,*

$$\lim_{n\to\infty} G^*\left(\frac{u}{n}, j\right) = \frac{1}{j!} \frac{1}{j} (u \wedge j)^j e^{-(u \wedge j)},$$

$$\lim_{n\to\infty} H^*\left(\frac{u}{n}\right) = \sum_{j=1}^{\infty} \frac{1}{j!} \frac{1}{j} (u \wedge j)^j e^{-(u \wedge j)},$$

and

$$\lim_{n\to\infty} p^*(j) = \frac{j^{j-1}}{j!} e^{-j} \equiv q^*(j).$$

THEOREM 5. *For $u \geqq 0$ and $j \geqq 1$,*

$$\lim_{n\to\infty} G_*\left(\frac{u}{n}, j\right) = \frac{(j-1)^{j-1}}{j^j} \sum_{l=j+1}^{\infty} \frac{1}{l!} u^l e^{-u} - \sum_{k=j+1}^{\infty} \binom{k-1}{j-1} \frac{(j-1)^{j-1}(k-j)^{k-j-1}}{k^k} \sum_{l=k}^{\infty} \frac{1}{l!}\left(\frac{k}{j} u\right)^l \times e^{-ku/j}$$

and

$$\lim_{n\to\infty} p_*(j) = \frac{(j-1)^{j-1}}{j^j} - \sum_{k=j+1}^{\infty} \binom{k-1}{j-1} \frac{(j-1)^{j-1}(k-j)^{k-j-1}}{k^k} \equiv q_*(j) .$$

THEOREM 6. *For $u \geqq 0$ and $j \geqq 1$,*

$$\lim_{n\to\infty} G_{**}\left(\frac{u}{n}, j\right) \doteq \frac{1}{j} \sum_{l=j}^{\infty} \frac{1}{l!} u^l e^{-u} - \sum_{k=j+1}^{\infty} \binom{k}{j} \frac{1}{k-1} \frac{j^j(k-j)^{k-j-1}}{k^k} \sum_{l=k}^{\infty} \frac{1}{l!}\left(\frac{k}{j} u\right)^l \times e^{-ku/j} ,$$

and

$$\lim_{n\to\infty} p_{**}(j) = \frac{1}{j} - \sum_{k=j+1}^{\infty} \binom{k}{j} \frac{1}{k-1} \cdot \frac{j^j(k-j)^{k-j-1}}{k^k} \equiv q_{**}(j) .$$

The following theorems interpret Theorems 4, 5, and 6 in terms of a Poisson limit theorem for the empirical df Γ_n and its (right continuous) inverse Γ_n^{-1}. Define processes N_n and N_n^{-1} by

$$N_n(t) = n\Gamma_n\left(\frac{t}{n}\right) \quad 0 \leqq t \leqq n , \qquad = t \quad n < t < \infty ,$$

and

$$N_n^{-1}(t) = n\Gamma_n^{-1}\left(\frac{t}{n}\right) \quad 0 \leqq t \leqq n , \qquad = t \quad n < t < \infty .$$

Let N denote a standard Poisson process (with parameter 1), and let N^{-1} denote the sum (right continuous inverse) process associated with N. The processes N_n, N_n^{-1}, $n \geqq 1$, N, and N^{-1} are all in $D[0, \infty)$, the class of functions on $[0, \infty)$ which are right continuous and have left-limits everywhere. It is well known that

(A) $\qquad N_n \Rightarrow N \quad$ and $\quad N_n^{-1} \Rightarrow N^{-1} \quad$ as $\quad N \to \infty$

where $\Rightarrow$ denotes weak convergence in the Skorohod topology on $D[0, T]$, for any $0 < T < \infty$. In this context, convergence of finite-dimensional distributions suffices for weak convergence (cf. Straf (1972), page 212; Miller (1976) has some related results). Also, it should be noted that the convergence of the inverse processes N_n^{-1} also holds (almost trivially) in the stronger uniform topology on $D[0, T]$ since all of the processes N_n^{-1} and N^{-1} have all their jumps at the integers.

But more is true (and does not follow directly from (A); it will be seen in the proof of Theorem 7 that what is needed is roughly that the convergence in (A) hold when T increases with n): by virtue of the strong law of large numbers

$$\lim_{t\to\infty} \frac{N(t)}{t} = 1 = \lim_{t\to\infty} \frac{N^{-1}(t)}{t} \quad \text{a.s.}$$

Thus the natural metrics for convergence to the processes N and N^{-1} in $D[0, \infty)$ involve the weight function

$$w(t) = 1/(t \vee 1)\,, \qquad t \in [0, \infty)\,.$$

Let B denote the subset of bounded functions in $D[0, \infty)$, and set

$$D_w = \{f \in D[0, \infty) : wf \in B\}\,.$$

Define metrics ρ_w and d_w on D_w as follows: For $f_1, f_2 \in D_w$ set

$$\rho_w(f_1, f_2) = \sup_{0\leq t<\infty} |f_1(t) - f_2(t)|w(t)\,.$$

Let Λ denote the class of strictly increasing continuous mappings of $[0, \infty)$ onto itself, and denote the identity function on $[0, \infty)$ by I: $I(t) = t$, $t \in [0, \infty)$. Denote the composition of $f \in D_w$ and $\lambda \in \Lambda$ by $f \circ \lambda$ and set

$$d_w(f_1, f_2) = \inf\{\varepsilon > 0 : \rho_w(\lambda, I) \leqq \varepsilon \text{ and } \rho_w(f_1, f_2 \circ \lambda) \leqq \varepsilon \text{ for some } \lambda \in \Lambda\}\,.$$

The metric d_w is one natural modification, for $D[0, \infty)$, of the Skorohod metric on $D[0, 1]$ (cf. Billingsley (1968), page 111 ff.); for other modifications, see Lindvall (1973), and Whitt (1972). Clearly $d_w(f_n, f) \to 0$ as $n \to \infty$ if and only if $\rho_w(f_n \circ \lambda_n, f) \to 0$ and $\rho_w(\lambda_n, I) \to 0$ as $n \to \infty$ for some sequence of functions $\lambda_n \in \Lambda$.

THEOREM 7. *There exist versions of the processes* $\{N_n\}_{n\geqq 1}$, N, *and* $\{N_n^{-1}\}_{n\geqq 1}$, N^{-1} *such that*

(B) $$d_w(N_n, N) \to_P 0 \quad \textit{as} \quad n \to \infty\,,$$

and

(C) $$\rho_w(N_n^{-1}, N^{-1}) \to_P 0 \quad \textit{as} \quad n \to \infty\,.$$

The proof of Theorem 7 relies on Theorem 1 of Chang (1964); it is related to some results of Runnenberg and Vervaat (1969) concerning the asymptotic independence and exponentiality of the sample spacings of uniform rv's.

Now let $E_1, E_2, \cdots$ denote the i.i.d. exponential (1) random variables associated with the processes N and N^{-1}, and set $Y_k = E_1 + \cdots + E_k = N^{-1}(k-)$ for

$k \geqq 1$. Define three pairs of rv's (Y^*, k^*), (Y_*, k_*), and (Y_{**}, k_{**}) by

$$\min_{1 \leq k < \infty} \frac{Y_k}{k} = \frac{Y_{k^*}}{k^*}, \qquad Y^* = Y_{k^*},$$

$$\max_{1 \leq k < \infty} \frac{Y_{k+1}}{k} = \frac{Y_{k_*+1}}{k_*}, \qquad Y_* = Y_{k_*+1},$$

$$\max_{1 \leq k < \infty} \frac{Y_k}{k} = \frac{Y_{k_{**}}}{k_{**}}, \qquad Y_{**} = Y_{k_{**}};$$

again all three pairs are a.s. defined uniquely. Also note that the strong law of large numbers (which guarantees that $k^{-1}Y_k \to 1$ a.s. as $k \to \infty$) together with the law of the iterated logarithm (which guarantees that oscillations about 1 occur a.s. so the approach to one is a.s. not monotone) imply that $P(k^* < \infty) = P(k_* < \infty) = P(k_{**} < \infty) = 1$.

Theorem 7 now implies the following reformulation of Theorems 4, 5, and 6:

THEOREM 8. *As* $n \to \infty$,

$$(nU^*, i^*) \to_d (Y^*, k^*),$$
$$(nU_*, i_*) \to_d (Y_*, k_*),$$

and,

$$(nU_{**}, i_{**}) \to_d (Y_{**}, k_{**});$$

hence the distributions of (Y^*, k^*), (Y_*, k_*), *and* (Y_{**}, k_{**}) *are given by Theorems* 4, 5, *and* 6 *respectively.*

The discrete distribution q^*, which is the distribution of k^* and the limiting distribution of i^*, is a Borel–Tanner distribution (cf. Haight and Breuer (1960)). The distributions q_* and q_{**} are apparently new. A table of the probabilities $q^*(j)$, $q_*(j)$, and $q_{**}(j)$ for $j = 1, \cdots, 20$ is given in Section 4.

The distribution of

$$\min_{1 \leq k < \infty} \frac{Y_k}{k} = \frac{Y^*}{k^*}$$

was found by Pyke (1959) (take $\lambda = 1$ in the expression at the top of page 571); also note that

$$P\left(\min_{1 \leq k < \infty} \frac{Y_k}{k} > \beta\right) = \lim_{n \to \infty} P(A_n(\beta)) = 1 - \beta$$

with the second equality holding trivially since $P(A_n(\beta)) = 1 - \beta$ for all $n \geqq 1$.

Pyke's Theorem 2 is related to $\max_{1 \leq k < \infty} (Y_{k+1}/k) = Y_*/k_*$. In that theorem Pyke studied $P(N(t) > \alpha t + x, 0 \leqq t \leqq T)$ where N denotes a Poisson process (with parameter 1 for our purposes). This probability is obviously 0 when $x = 0$, but the probability $P(N(t) > \alpha t, Y_1 \leqq t \leqq T)$, where Y_1 is the first jump point of the process, is not zero and in fact

$$P(N(t) > \alpha t, Y_1 \leqq t < \infty) = P\left(\max_{1\leqq k<\infty} \frac{Y_{k+1}}{k} < \alpha^{-1}\right)$$
$$= \lim_{n\to\infty} P\left(\max_{1\leqq i\leqq n-1} \frac{nU_{i+1}}{i} < \alpha^{-1}\right)$$
$$= 1 - e^{-1/\alpha} - \sum_{k=1}^{\infty} \frac{(k-1)^{k-1}}{k!\,\alpha^k} e^{-k/\alpha} ;$$

this distribution was found by Chang (1964).

3. Proofs. Theorem 1 may be found on page 597 of Dempster (1959); related results are contained in Dwass (1959) and Eicker (1970). Theorem 2 is proved by using the conditional independence of the first j and the last $n - j - 1$ order statistics given U_{j+1} (cf. Rényi (1973), page 291) to write

$$G_*(u, j) = \int_{(0,u]} P(i_* = j \mid U_{j+1} = v) P(U_{j+1} \in dv)$$
$$= \int_{(0,u]} P\left(\Gamma_n(t) \geqq \frac{j}{nv} t,\ U_1 \leqq t \leqq \frac{nv}{j} \wedge 1 \,\middle|\, U_{j+1} = v\right) P(U_{j+1} \in dv)$$
$$(1) \qquad = \int_{(0,u]} P\left(\Gamma_n(t) \geqq \frac{j}{nv} t,\ U_1 \leqq t \leqq v \mid U_{j+1} = v\right)$$
$$\times P\left(\Gamma_n(t) \geqq \frac{j}{nv} t,\ v \leqq t \leqq \frac{nv}{j} \wedge 1 \,\middle|\, U_{j+1} = v\right) P(U_{j+1} \in dv)$$
$$\equiv \int_{(0,u]} P_1 P_2 P(U_{j+1} \in dv) .$$

The conditional probability P_1 may be evaluated using a result of Chang ((1964), page 29):

$$P_1 = P\left(\Gamma_n(t) \geqq \frac{j}{nv} t,\ U_1 \leqq t \leqq v \mid U_{j+1} = v\right)$$
$$(2) \qquad = P(\Gamma_j(t) \geqq t,\ U_1 \leqq t \leqq 1)$$
$$= P\left(\inf_{\Gamma_j(t)>0} \frac{\Gamma_j(t)}{t} = 1\right) = (j-1)^{j-1}/j^j .$$

To evaluate the conditional probability P_2 first note that when $j = n - 1$, $P_2 = 1$ for all $0 \leqq v \leqq 1$. For $j \leqq n - 2$, P_2 may be evaluated by means of results of Dempster (1959) or Eicker (1970) depending on whether $v \geqq j/n$ or $v \leqq j/n$: by replacing t by $1 - t$, using $(U_1, \cdots, U_n) =_d (1 - U_n, \cdots, 1 - U_1)$, and rescaling, one obtains

$$P_2 = P\left(\Gamma_n(t) \geqq \frac{j}{nv} t,\ v \leqq t \leqq \frac{nv}{j} \wedge 1 \,\middle|\, U_{j+1} = v\right)$$
$$= P\left(\Gamma_{n-j-1}(t) \leqq \delta + \frac{1-\delta}{1-\varepsilon} t,\ \frac{1 - (nv/j \wedge 1)}{1-v} \leqq t \leqq 1\right)$$
$$= P\left(\Gamma_{n-j-1}(t) \leqq \delta + \frac{1-\delta}{1-\varepsilon} t,\ 0 \leqq t \leqq 1\right), \qquad v \geqq \frac{j}{n}$$
$$= P\left(\Gamma_{n-j-1}(t) \leqq \delta + \frac{1-\delta}{1-\varepsilon} t,\ \frac{j - nv}{j(1-v)} \leqq t \leqq 1\right), \qquad v \leqq \frac{j}{n} ,$$

where

$$\delta = \left(1 - \frac{j}{nv}\right)\Big/\left(1 - \frac{j+1}{n}\right), \qquad \frac{1-\delta}{1-\varepsilon} = \frac{j}{nv}(1-v)\Big/\left(1 - \frac{j+1}{n}\right).$$

In the case $v \geqq j/n$ ($\delta \geqq 0$), the probability may be evaluated using equation (5′) of Dempster (1959):

$$\begin{aligned} (3) \qquad & P\left(\Gamma_{n-j-1}(t) \leqq \delta + \frac{1-\delta}{1-\varepsilon}t,\ 0 \leqq t \leqq 1\right) \\ &= 1 - \left\{\frac{v}{j(1-v)}\right\}^{n-j-1} \sum_{i=0}^{j_v} \binom{n-j-1}{i}(i+1)^{i-1} \times \left(\frac{j(1-v)}{v} - 1 - i\right)^{n-j-1-i} \\ &= 1 - \left\{\frac{v}{j(1-v)}\right\}^{n-j-1} \sum_{i=0}^{n-j-2} \binom{n-j-1}{i}(i+1)^{i-1} \times \left(\frac{j(1-v)}{v} - 1 - i\right)^{n-j-1-i} 1_{(0,j/(j+i+1))}(v) \end{aligned}$$

where j_v is the largest integer strictly smaller than $(j(1-v)/v) - 1$ and $n-j-1$. Note that although the argument above required $v \geqq j/n$, the resulting expression in well defined for $0 < v \leqq j/n$ as well (with all $n-j-2$ terms of the sum entering for $v \in (0, j/(n-1))$. We now show that the above expression also gives the probability P_2 for $v \leqq j/n$.

In the case $v \leqq j/n$ ($\delta \leqq 0$), the probability P_2 may be evaluated using equation (1.4) of Eicker (1970): by writing

$$\alpha = \frac{j - nv}{j(1-v)}, \qquad \beta = 1 - \frac{v}{j(1-v)}, \qquad \gamma = \frac{1-\delta}{1-\varepsilon},$$

one obtains, since $\delta + \gamma\alpha = 0$, and $\delta + \gamma\beta = 1$,

$$\begin{aligned} P(\Gamma_{n-j-1}(t) \leqq \delta + \gamma t, \alpha \leqq t \leqq 1) &= P(\Gamma_{n-j-1}(t) \leqq \delta + \gamma t, \alpha \leqq t \leqq \beta) \\ &= (1 - \beta + \gamma^{-1})^{n-j-2}(1-\beta) \\ &= \left\{\frac{v}{j(1-v)}\right\}^{n-j-1} (n-j)^{n-j-2}. \end{aligned}$$

That this equals the last expression in (3) for $0 < v \leqq j/n$ may be seen by means of the following Abel-type identity (cf. Eicker (1970), page 2081, (1.21)): for any real number b,

$$\sum_{i=0}^{k} \binom{k}{i}(i+1)^{i-1}(b-1-i)^{k-i} = b^k.$$

The asserted identity follows by choosing $k = n-j-1$, $b = j(1-v)/v$, and rearranging. Hence, in all cases (recall the convention following Theorem 2) the conditional probability P_2 is given by the last line of (3).

Finally, G_* and p_* are obtained from (1), (2), and (3) by straightforward integration:

$$G_*(u, j) = \int_{(0,u]} P_1 P_2 P(U_{j+1} \in dv) \tag{4}$$
$$= \frac{(j-1)^{j-1}}{j^j} P(U_{j+1} \leqq u) - \frac{n!\,(j-1)^{j-1}}{j!\,(n-j-1)!\,j^{n-1}} \sum_{i=0}^{n-j-2} A_{ji}$$

where

$$A_{ji} = \binom{n-j-1}{i}(i+1)^{i-1} \int_0^u v^{n-1}\left(\frac{j(1-v)}{v} - 1 - i\right)^{n-j-1-i} 1_{(0,j/(j+i+1))}(v)\,dv\,.$$

Now

$$P(U_{j+1} \leqq u) = P(B(n, u) \geqq j+1) = \sum_{l=j+1}^{n} \binom{n}{l} u^l (1-n)^{n-l}\,,$$

where $B(n, u)$ denotes a binomial rv with parameters n and u, and, for $0 \leqq i \leqq n-j-2$,

$$\int_0^u v^{n-1}\left(\frac{j(1-v)}{v} - 1 - i\right)^{n-j-1-i} 1_{(0,j/(j+i+1))}(v)\,dv$$
$$= \frac{j^n (j+i)!\,(n-j-1-i)!}{(j+i+1)^{j+i+1} n!} \sum_{l=j+i+1}^{n} \binom{n}{l} \left(\frac{j+i+1}{j} u \wedge 1\right)^l$$
$$\times \left(1 - \frac{j+i+1}{j} u \wedge 1\right)^{n-l}.$$

Upon using these last expressions to replace $P(U_{j+1} \leqq u)$ and A_{ji} in (4) and changing variables of summation by setting $k = j + i + 1$ one obtains $G_*(u, j)$ as given in Theorem 2. Letting $u \to 1$ in this expression yields the probability $p_*(j)$ given in Theorem 2. That the $p_*(j)$'s sum to one may be seen by reversing the order of summation of the second term and using the identity (cf. Lemma 2 of Birnbaum and Pyke (1958))

$$\sum_{j=1}^{k-1} \binom{k-1}{j-1} (j-1)^{j-1} (k-j)^{k-j-1} = (k-1)^{k-1}\,.$$

The proof of Theorem 3 is similar to the proof of Theorem 2 given above, and therefore we omit many of the details. Here and in the proof of Theorem 7 it will be convenient to denote the order statistics of the sample of n Uniform (0, 1) rv's by $0 \equiv U_{n,0} < U_{n,1} < \cdots < U_{n,n} < U_{n,n+1} \equiv 1$. Again the conditional independence of the first $j-1$ and last $n-j$ order statistics given $U_{n,j}$ is used to write, for $j \in \{1, \cdots, n\}$,

$$G_{**}(u, j) = \int_{(0,u]} P(i_{**} = j \mid U_{n,j} = v) P(U_{n,j} \in dv)$$
$$= \int_{(0,u]} P\left(U_{n,i} \leqq \frac{v}{j} i,\ i = 1, \cdots, j-1, j+1, \cdots, n \,\middle|\, U_{n,j} = v\right)$$
$$\times P(U_{n,j} \in dv) \tag{5}$$
$$= \int_{(0,u]} P\left(U_{n,i} \leqq \frac{v}{j} i,\ i = 1, \cdots, j-1 \,\middle|\, U_{n,j} = v\right)$$
$$\times P\left(U_{n,i} \leqq \frac{v}{j} i,\ i = j+1, \cdots, n \,\middle|\, U_{n,j} = v\right) \cdot P(U_{n,j} \in dv)$$
$$\equiv \int_{(0,u]} P_1' P_2' P(U_{n,j} \in dv)$$

where, for $j = 1$, $P_1' \equiv 1$; and, for $j = n$, $P_2' \equiv 1$.

Now, for $j > 1$,

$$P_1' = P\left(U_{n,i} \leqq \frac{v}{j} i, i = 1, \cdots, j-1 \,\middle|\, U_{n,j} = v\right)$$
$$= P\left(U_{j-1,i} \leqq \frac{i}{j}, \; i = 1, \cdots, j-1\right)$$
$$= \frac{1}{j}$$

from Rényi (1973) (page 294, (2.20)) and a combinatorial identity, or, alternatively, by noting that $U_{j-1,i} = \sum_{k=1}^{i} D_k$ where $D_k = U_{j-1,k} - U_{j-1,k-1}$ are symmetrically dependent, and hence by a result of Anderson (1953) (see also Pyke (1959), page 575),

$$P_1' = P\left(U_{j-1,i} \leqq \frac{i}{j}, \; i = 1, \cdots, j-1\right)$$
$$= P\left(\textstyle\sum_{k=1}^{i}\left(D_k - \frac{1}{j}\right) \leqq 0, \; i = 1, \cdots, j-1\right) \tag{6}$$
$$= \frac{1}{j}.$$

The computation of P_2' is similar to that of P_2 above, again using the results of Dempster and Eicker; some manipulation yields

$$P_2' = P\left(U_{n,i} \leqq \frac{v}{j} i, i = j+1, \cdots, n \,\middle|\, U_{n,j} = v\right)$$
$$= P\left(U_{n-j,i} \geqq \left(\frac{i}{n-j} - \delta\right)\frac{1-\varepsilon}{1-\delta}, \; i = 1, \cdots, n-j\right) \tag{7}$$
$$= 1 - \left\{\frac{v}{j(1-v)}\right\}^{n-j} \textstyle\sum_{i=0}^{n-j-1} \binom{n-j}{i}(i+1)^{i-1}\left(\frac{j(1-v)}{v} - 1 - i\right)^{n-j-i}$$
$$\times 1_{(0, j/(j+i+1))}(v)$$

where

$$\delta = \frac{(n+1)v - j}{(n-j)v}, \qquad \frac{1-\delta}{1-\varepsilon} = \frac{(n-j)v}{(1-v)j}.$$

G_{**} and p_{**} as given in Theorem 3 are obtained from (5), (6), and (7) by straightforward integration and letting $u \to 1$; again Lemma 2 of Birnbaum and Pyke (1958) may be used to show that the $p_{**}(j)$'s sum to one; we omit the details.

Theorems 4, 5, and 6 follow directly from Theorems 1, 2, and 3 respectively by application of Stirling's formula. Note that the terms in the expression for $q_*(j)(q_{**}(j))$ are asymptotic to $((j-1)^{j-1}e^{-j}/(j-1)!)k^{-2}((j^j e^{-j}/j!)k^{-2})$ as $k \to \infty$, and hence the sums converge.

PROOF OF THEOREM 7. We first prove (C). Suppose $\{T_n\}$ is a sequence of integers such that $T_n \to \infty$ and $n^{-1}T_n(\log n)^2 \to 0$ as $n \to \infty$. Then

$$\begin{aligned}\rho_w(N_n^{-1}, N^{-1}) &= \sup_{0 \leqq t < \infty} |N_n^{-1}(t) - N^{-1}(t)|w(t) \\ &\leqq |nU_{n,1} - Y_1| + \max_{1 \leqq k \leqq T_n} \left| \frac{nU_{n,k+1}}{k} - \frac{Y_{k+1}}{k} \right| \\ &\quad + \max_{T_n < k \leqq n-1} \left| \frac{nU_{n,k+1}}{k} - 1 \right| + \max_{T_n < k < \infty} \left| \frac{Y_{k+1}}{k} - 1 \right| \\ &\equiv R_1 + R_2 + R_3 + R_4 .\end{aligned}$$

By the strong law of large numbers $R_4 \to 0$ a.s. as $n \to \infty$ and $T_n \to \infty$ in any way. Using Theorem 1 of Chang (1955) (cf. (4), page 20) together with $T_n \to \infty$, and setting

$$M_n \equiv \max_{1 \leqq k \leqq n-1} (nU_{n,k+1}/k) ,$$

$$\begin{aligned}R_3 &= \max_{T_n < k \leqq n-1} \frac{nU_{n,k+1}}{k+1} \left| \frac{k+1}{k} - \frac{k+1}{nU_{n,k+1}} \right| \\ &\leqq M_n \left\{ o(1) + \sup_{(T_n/n) < \Gamma_n(t) \leqq 1} \left| 1 - \frac{\Gamma_n(t)}{t} \right| \right\} \\ &= O_P(1) o_P(1) = o_P(1) \qquad \text{as} \quad n \to \infty .\end{aligned}$$

To handle R_1 and R_2, set $D_{ni} = n(U_{n,i} - U_{n,i-1})$, $i = 1, \cdots, n+1$ and consider the explicit construction of the D_{ni}, and hence the processes Γ_n, N_n, Γ_n^{-1}, and N_n^{-1}, as (cf. Pyke (1965), Section 4.1)

$$D_{ni} \equiv n \frac{E_i}{Y_{n+1}} = n \frac{E_i}{E_1 + \cdots + E_{n+1}} , \qquad i = 1, \cdots, n+1 .$$

Then,

$$\begin{aligned}R_1 + R_2 &= |D_{n1} - E_1| + \max_{1 \leqq k \leqq T_n} \left| \frac{1}{k} \sum_{i=1}^{k+1} (D_{ni} - E_i) \right| \\ &\leqq 3 \max_{1 \leqq i \leqq T_n + 1} |D_{ni} - E_i| \\ &\to_P 0 \qquad \text{as} \quad n \to \infty ,\end{aligned}$$

since, letting $c_n = 1 + \log n$ for $n \geqq 1$, for any $\varepsilon > 0$,

$$\begin{aligned}P(|D_{ni} - E_i| > \varepsilon) &= P\left(E_i \left| \frac{n}{Y_{n+1}} - 1 \right| > \varepsilon \right) \\ &\leqq P\left(\left| \frac{Y_{n+1}}{n} - 1 \right| > \frac{\varepsilon}{2c_n} \right) + P(E_i > c_n) \\ &= O(n^{-1}(\log n)^2) ,\end{aligned}$$

and hence,

$$\begin{aligned}P(\max_{1 \leqq i \leqq T_n + 1} |D_{ni} - E_i| > \varepsilon) &\leqq \sum_{i=1}^{T_n+1} P(|D_{ni} - E_i| > \varepsilon) \\ &= (T_n + 1) O(n^{-1} (\log n)^2) \\ &\to 0 \qquad \text{as} \quad n \to \infty .\end{aligned}$$

Alternatively, $R_1 + R_2$ could be handled by using the results of Runnenberg and Vervaat (1969), which give the asymptotic independence and exponentiality of the first T_n uniform spacings when $T_n \to \infty$ and $n^{-1}T_n \to 0$ as $n \to \infty$.

To prove (B), define a sequence of (random) functions λ_n, $n \geqq 1$, in $D[0, \infty)$ by

$$\lambda_n(Y_k) = nU_{n,k}, \qquad k = 0, \cdots, n+1$$

with λ_n linear between these points for $0 \leqq t \leqq Y_{n+1}$; and set $\lambda_n(t) = n + (t - Y_{n+1})$ for $Y_{n+1} < t < \infty$. Here $Y_k = N^{-1}(k-) = E_1 + \cdots + E_k$. That $\lambda_n \in \Lambda$ a.s. for all $n \geqq 1$ is clear. From the discussion preceding Theorem 7, it suffices to prove

$$\rho_w(N_n \circ \lambda_n, N) \to_P 0 \quad \text{as} \quad n \to \infty, \tag{8}$$

and

$$\rho_w(\lambda_n, I) \to_P 0 \quad \text{as} \quad n \to \infty. \tag{9}$$

But (8) follows easily since $N_n(\lambda_n(t)) = N(t)$ for $0 \leqq t < Y_{n+1}$, and hence

$$\begin{aligned} \rho_w(N \circ \lambda_n, N) &= \sup_{Y_{n+1} \leqq t < \infty} |\lambda_n(t) - N(t)| w(t) \\ &\leqq \sup_{Y_{n+1} \leqq t < \infty} |1 - t^{-1}N(t)| + \left| \frac{n}{Y_{n+1}} - 1 \right| \\ &\to 0 \quad \text{a.s.} \quad \text{as} \quad n \to \infty, \end{aligned}$$

since $Y_{n+1} \to \infty$ a.s. as $n \to \infty$, and $t^{-1}N(t) \to 1$ a.s. as $t \to \infty$, both by the strong law of large numbers.

The proof of (9) proceeds in much the same way as the proof of (C) given above, using the same construction of the Uniform spacings, and hence we omit many of the details: by a similar argument,

$$\begin{aligned} \rho_w(\lambda_n, I) &\leqq \max_{1 \leqq k \leqq n+1} |nU_{n,k} - Y_k| / Y_k \\ &\leqq \max_{1 \leqq k \leqq n+1} \left| \frac{nU_{n,k}}{k} - \frac{Y_k}{k} \right| \left\{ \min_{1 \leqq k < \infty} \frac{Y_k}{k} \right\}^{-1} \\ &= o_P(1) O_P(1), \qquad n \to \infty, \end{aligned}$$

and this completes the proof of Theorem 7. □

Theorem 8 follows directly from Theorem 7 and the a.s. uniqueness of the rv's k^*, k_*, and k_{**}; we omit the proof.

4. The probabilities q^*, q_*, and q_{**}. Table 1 gives the values of q^*, q_*, and q_{**} for $j = 1, \cdots, 20$.

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TABLE 1

j	$q^*(j)$	$q_*(j)$	$q_{**}(j)$
1	.3679	.5430	.2308
2	.1353	.1069	.1106
3	.0747	.0543	.0697
4	.0488	.0345	.0495
5	.0351	.0245	.0378
6	.0268	.0185	.0302
7	.0213	.0147	.0249
8	.0175	.0120	.0210
9	.0146	.0101	.0181
10	.0125	.0086	.0158
11	.0109	.0075	.0139
12	.0095	.0066	.0125
13	.0085	.0058	.0112
14	.0076	.0052	.0102
15	.0068	.0047	.0093
16	.0062	.0043	.0085
17	.0057	.0039	.0079
18	.0052	.0036	.0073
19	.0048	.0033	.0068
20	.0044	.0031	.0063
> 20	.1759	.1249	.2977

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