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A LAW OF THE ITERATED LOGARITHM FOR FUNCTIONS OF ORDER STATISTICS

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A general law of the iterated logarithm for linear combinations of order statistics is proved. The key tools are (1) iterated logarithm convergence of the uniform empirical process U_n in ρ_q-metrics due to B. R. James and (2) almost sure "nearly linear" bounds for the empirical distribution function. A law of the iterated logarithm for the quantile process is also established.

1. Introduction. In [16] Shorack proved general central limit theorems for linear combinations of order statistics. Our purpose here is to establish a law of the iterated logarithm for order statistics (Theorem 4) which parallels Shorack's Theorem 1. Our proof relies upon (1) the iterated logarithm convergence of the uniform empirical process with respect to ρ_q-metrics due to James [11] (or Theorem 2 of [20] with a simpler proof), and (2) almost sure "nearly linear" bounds for the uniform empirical df. These bounds are proved in Section 2.

We adopt the notation of [16]: $\xi_1, \cdots, \xi_n$ are i.i.d. uniform (0, 1) rv's with empirical df Γ_n and $0 \leqq \xi_{n1} \leqq \cdots \leqq \xi_{nn} \leqq 1$ denote the order statistics of the sample. The *uniform empirical process* U_n is the process on [0, 1] defined by $U_n = n^{\frac{1}{2}}(\Gamma_n - I)$ where I denotes the identity function, $I(t) = t$; the *uniform quantile process* V_n on [0, 1] is defined by $V_n = n^{\frac{1}{2}}(\Gamma_n^{-1} - I)$ where Γ_n^{-1} is the left continuous inverse of Γ_n. Theorem 4 below presents a law of the iterated logarithm for

$$T_n \equiv n^{-1} \sum_{i=1}^{n} c_{ni}\, g_n(\xi_{ni}) + \sum_{k=1}^{\kappa} d_{nk}\, g_n(\xi_{n,[np_k]+1}) \tag{1}$$

where, for $n \geqq 1$, $c_{n1}, \cdots, c_{nn}$ are known constants, and g_n, for each n, is in the class $\mathscr{G}$ of left continuous functions on (0, 1) that are of bounded variation on $(\theta, 1 - \theta)$ for all $\theta > 0$. Here $d_{n1}, \cdots, d_{n\kappa}$ for $n \geqq 1$ are known constants associated with the points $0 < p_1 < \cdots < p_\kappa < 1$. For emphasis we repeat Remark 1 of [16]:

REMARK 1. If $g_n = h(F_n^{-1})$ for some sequence of df's F_n in the class $\mathscr{F}$ of all df's, then T_n has the same distribution as does $n^{-1} \sum_1^n c_{ni} h(X_{ni}) + \sum_1^\kappa d_{nk} h(X_{n,[np_k]+1})$ where $X_{n1} \leqq \cdots \leqq X_{nn}$ are the order statistics of a random sample of size n from F_n.

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As mentioned above, our approach in Section 4 to a log-log law for T_n is by way of a log-log law for the process U_n. A law of the iterated logarithm for T_n could also be approached by the projection method used by Stigler [17], but our method has advantages for unbounded score functions J. In the case of bounded J, $\kappa = 0$, and fixed continuous df F ($g_n = g = F^{-1}$ for all $n \geqq 1$ in our notation), Ghosh [8] has established a law of the iterated logarithm for T_n using the methods of Moore [14]. Theorem 4 gives additional information even in this special case.

2. Almost sure "nearly linear" bounds for the empirical df. We begin by establishing almost sure "nearly linear" bounds (Theorem 1) for the empirical df Γ_n and its left continuous inverse Γ_n^{-1}. These bounds play an important role in our proofs in Sections 3 and 4 and are the appropriate analogue, for our purposes, of the linear bounds of Lemma A3 of [16]. We obtain one set of these nearly linear bounds by use of the techniques of Theorem 1 of [20]. The other set of bounds is established via exponential inequalities for the tail probabilities of centered beta rv's due to Albers, Bickel and van Zwet [1], and Wellner [19]. The exponential inequalities for probabilities then yield bounds for the central absolute moments of the same beta rv's. These inequalities improve on moment inequalities for beta rv's established by Bickel [3], Blom [5], and van Zwet [22].

Let $0 \leqq \xi_{n1} \leqq \cdots \leqq \xi_{nn} \leqq 1$ denote the order statistics of the sample $\xi_1, \cdots, \xi_n$ of uniform (0, 1) rv's. Then ξ_{ni} has a beta $(i, n-i+1)$ distribution with mean $p_i \equiv i/(n+1) \equiv 1 - q_i$ and variance $p_i q_i/(n+2)$.

LEMMA 1. *For all* $n \geqq 1$, $1 \leqq i \leqq n$, *and* $\lambda \geqq 1$

$$P(n^{\frac{1}{2}}|\xi_{ni} - p_i| \geqq (p_i q_i)^{\frac{1}{2}}\lambda) \leqq 2e^{-\lambda/5}. \tag{2}$$

PROOF. Lemma A2.1 of [1] (page 148) gives the inequality

$$P(n^{\frac{1}{2}}|\xi_{ni} - p_i| \geqq (p_i q_i)^{\frac{1}{2}}\lambda) \leqq 2\exp(-3\lambda^2/(6\lambda + 8))$$

for $\lambda \geqq 0$; since $3\lambda^2/(6\lambda+8) \geqq \lambda/5$ for $\lambda \geqq \frac{8}{9}$, (2) holds. Essentially the same inequality was established independently in [19]. The proofs in both [1] and [19] involve rewriting the probability of (2) in terms of binomial rv's and then using exponential inequalities due to Hoeffding [10]. □

LEMMA 2. *For all real* $r > 0$, $n \geqq 1$, *and* $1 \leqq i \leqq n$

$$E|\xi_{ni} - p_i|^r \leqq C_r(p_i q_i/n)^{r/2} \leqq C_r(i/n^2)^{r/2} \tag{3}$$

where $C_r \equiv 1 + 2 \cdot 5^r \cdot \Gamma(r+1)$.

PROOF. Let $h = \frac{1}{5}$. The exponential inequality (2) implies that

$$1 - F(s) \equiv P(|\xi_{ni} - p_i| \geqq s) \leqq 2\exp(-hs/s_0)$$

for $s \geqq s_0 \equiv (p_i q_i/n)^{\frac{1}{2}}$. Hence

$$\begin{aligned} E|\xi_{ni} - p_i|^r &= \int_0^\infty t^r \, dF(t) = r \int_0^\infty (1 - F(s)) s^{r-1} \, ds \\ &\leqq r \int_0^{s_0} s^{r-1} \, ds + 2r \int_{s_0}^\infty \exp(-hs/s_0) s^{r-1} \, ds \\ &\leqq (1 + 2r\Gamma(r)h^{-r}) s_0^r \\ &= C_r s_0^r . \end{aligned}$$

□

Before proceeding to the application of (3) in Theorem 1, we compare it with known inequalities for the central moments of beta rv's. Blom [5] obtains $E|\xi_{ni} - p_i|^r \leqq M_r n^{-r/2}$ with M_r independent of i and n. This inequality follows easily from (3), but is insufficient for our purposes in Theorem 1. The results of Bickel [3] and van Zwet [22] apply when $n \to \infty$ and $i/n \to \alpha$ with $0 < \alpha < 1$. Bickel obtains $E(\xi_{ni} - p_i)^r = C_r'(p_i q_i/n)^{r/2} + o(n^{-r/2})$ and van Zwet gives $E(\xi_{ni} - p_i)^r = C_r''(p_i q_i/n)^{r/2} + O(n^{-r/2-1})$. These results will not suffice to prove our Theorem 1 since we require the inequality for $1 \leqq i \leqq n$. Now we use (3) to establish almost sure "nearly linear" bounds for Γ_n and Γ_n^{-1}.

THEOREM 1. *Let $\tau_1, \tau_2 > 1$ be fixed. Then there exists $0 < \beta = \beta(\tau_1, \tau_2) < \frac{1}{2}$ and a set $A \subset \Omega$ with $P(A) = 1$ having the following properties: for all $\omega \in A$ there is an $N \equiv N(\omega, \tau_1, \tau_2)$ for which $n \geqq N$ implies*

$$(4) \qquad 1 - \left(\frac{1-t}{\beta}\right)^{1/\tau_2} \leqq \Gamma_n(t) \leqq (t/\beta)^{1/\tau_1} \quad \text{for} \quad 0 \leqq t \leqq 1 ,$$

$$(5) \qquad \beta t^{\tau_1} \leqq \Gamma_n(t) \quad \text{for all} \quad t \quad \text{such that} \quad 0 < \Gamma_n(t) ,$$

$$(6) \qquad \Gamma_n(t) \leqq 1 - \beta(1-t)^{\tau_2} \quad \text{for all} \quad t \quad \text{such that} \quad \Gamma_n(t) < 1 ,$$

$$(7) \qquad \beta t^{\tau_1} \leqq \Gamma_n^{-1}(t) \leqq 1 - \beta(1-t)^{\tau_2} \quad \text{for} \quad 0 \leqq t \leqq 1 ,$$

$$(8) \qquad \Gamma_n^{-1}(t) \leqq (t/\beta)^{1/\tau_1} \quad \text{for} \quad t \geqq \frac{1}{n} , \qquad \text{and}$$

$$(9) \qquad 1 - \left(\frac{1-t}{\beta}\right)^{1/\tau_2} \leqq \Gamma_n^{-1}(t) \quad \text{for} \quad t \leqq 1 - \frac{1}{n} .$$

PROOF. (Braun [6] established (5) for $\tau_1 > \frac{8}{7}$ by an argument involving direct computation of eighth moments.) Note that it suffices to prove only the upper bound of (4) and (8): by replacing ξ_i by $1 - \xi_i$, by interchanging τ_1 and τ_2, and by use of symmetry about the identity function or 45 degree line, the upper bound of (4) implies the remaining inequalities in (4) and (7); similarly, (8) implies the remaining inequalities (5), (6) and (9).

To prove (8), let $\rho = \tau_1^{-1}$, $c > 0$, $\xi_{ni}^* \equiv \xi_{ni} - p_i$, and define

$$A_n \equiv \left\{ \max_{1 \leqq i \leqq n} \frac{|\xi_{ni}^*|}{c(i/n)^\rho} \geqq 1 \right\} .$$

Choose an integer $k > 2$ so large that $(2k-1)/2k > \rho$. Then by the Birnbaum–Marshall inequality [4] and the fact that $\{\xi_{ni}^*/(n-i+1), 1 \leqq i \leqq n\}$ is a martingale,

$$P(A_n) \leqq \textstyle\sum_1^n (a_{ni}^{2k} - a_{ni+1}^{2k}) E|\xi_{ni}^*/(n-i+1)|^{2k}$$

where $a_{ni} \equiv [(n-i+1)/c](n/i)^\rho$, $1 \leqq i \leqq n$, and $a_{nn+1} \equiv 0$. Hence, using (3) to obtain the second inequality and choosing $0 < \varepsilon < (2k-1-2k\rho) \wedge (k-2)$, we have

$$\begin{aligned} P(A_n) &\leqq \textstyle\sum_1^n a_{ni}^{2k} E|\xi_{ni}^*/(n-i+1)|^{2k} \\ &\leqq c^{-2k} C_{2k} \textstyle\sum_1^n (n/i)^{2k\rho}(i/n^2)^k \\ &\leqq c^{-2k} C_{2k} \textstyle\sum_1^n (1/n^{2+\varepsilon})(i/n)^{2k-2-\varepsilon}(i/n)^{-2k\rho} \\ &\leqq c^{-2k} C_{2k} n^{-1-\varepsilon} \textstyle\int_0^1 s^{-\gamma}\, ds \end{aligned}$$

with $\gamma \equiv 2k\rho - 2k + 2 + \varepsilon < 1$. Therefore $\sum_1^\infty P(A_n) < \infty$ and $P(A_n \text{ i.o.}) = 0$. Hence, for $n \geqq N(\omega, \rho)$ we have

$$|\xi_{ni}^*| \leqq c(i/n)^\rho$$

for $1 \leqq i \leqq n$ which implies that

$$\begin{aligned} \Gamma_n^{-1}\left(\frac{i}{n}\right) = \xi_{ni} &\leqq i/(n+1) + c(i/n)^\rho \\ &\leqq (1+c)(i/n)^\rho \end{aligned}$$

for $1 \leqq i \leqq n$, and this in turn yields

$$\Gamma_n^{-1}\left(\frac{i}{n}\right) \leqq 2^\rho(1+c)((i-1)/n)^\rho$$

for $2 \leqq i \leqq n$, $n \geqq N(\omega, \rho)$ and all ω in a set with probability one. This proves (8) with $\beta \equiv 2^{-1}(1+c)^{-\tau_1}$.

To prove the upper bound of (4), again let $\rho = \tau_1^{-1}$, $c > 0$, but now define $\Gamma_n^* = \Gamma_n - I$ and

$$B_n = \left\{ \sup_{0<t\leqq 1} \frac{|\Gamma_n^*(t)|}{ct^\rho} \geqq 1 \right\}.$$

Application of the methods of Theorem 1 of [20] (with the inequality of Lemma 2 there replaced by a moment inequality) yields, for any $1 < \gamma < 1/\rho < 2$,

$$P(B_n) \leqq (2/c)^\gamma E|D_n|^\gamma$$

where $D_n = (1/n) \sum_1^n Y_i$, and, with $h(s) \equiv s^{-\rho}$,

$$Y_i = h(\xi_i) - \textstyle\int_0^{\xi_i} (1-I)^{-1} h \, dI$$

are i.i.d. with mean zero and $E|Y_1|^\gamma < \infty$. Hence, using an inequality due to von Bahr and Esseen [2], we find that

$$P(B_n) \leqq 2(2/c)^\gamma E|Y_1|/n^{\gamma-1};$$

this implies that $\sum_{n=1}^\infty P(B_{[n^\eta]}) < \infty$ where η satisfies $\eta > (\gamma-1)^{-1} > \rho/(1-\rho)$. Finally, (the Banach space version of) Skorohod's inequality may be applied to gain control for sample sizes between points of the subsequence n^η, yielding

$P(B_n \text{ i.o.}) = 0$ (with c replaced by $c' = 2^{\gamma+1}c$). Hence for $n \geqq N(\omega, \rho)$,

$$|\Gamma_n^*(t)| \leqq c' t^\rho, \qquad 0 \leqq t \leqq 1$$

or,

$$\Gamma_n(t) \leqq (1 + c')t^\rho, \qquad 0 \leqq t \leqq 1.$$

This implies that for $n \geqq N(\omega, \rho)$ and all ω in a set with probability one

$$\Gamma_n(t) \leqq (t/\beta)^\rho, \qquad 0 \leqq t \leqq 1$$

where $0 < \beta \equiv 2^{-1}(1 + c')^{-\rho} < \frac{1}{2}$, proving the upper bound of (4). □

Observe that neither τ_1 nor τ_2 may be taken equal to one in Theorem 1: the results of Kiefer [13] and Robbins and Siegmund [15] imply that both

$$\limsup_{n\to\infty} \sup_{0<t\leq 1} \frac{\Gamma_n(t)}{t} = +\infty$$

and

$$\limsup_{n\to\infty} \sup_{1/n\leq t\leq 1} \frac{\Gamma_n^{-1}(t)}{t} = +\infty$$

with probability one. In [20a] we have obtained an integral test for upper almost sure bounding curves (the upper bound of (4)).

3. A law of the iterated logarithm for the quantile process. To handle the term

$$\sum_1^K d_{nk}\, g_n(\xi_{n,[np_k]+1})$$

of T_n or to establish a log-log version of Theorem 2 of [16] we need a law of the iterated logarithm for the uniform quantile process $V_n = n^{\frac{1}{2}}(\Gamma_n^{-1} - I)$. For our purposes in Theorem 4 of the following section, convergence with respect to the supremum metric ρ more than suffices. This conclusion follows easily from the results of Finkelstein [7] concerning the log-log convergence of $U_n = n^{\frac{1}{2}}(\Gamma_n - I)$ with respect to ρ together with Theorem 2 of Kiefer [12], or Lemma 1 of Vervaat [18], or the identity (10) below. Convergence with respect to the metric ρ will not, however, suffice for a log-log version of Theorem 2 of [16]; this type of result requires the convergence of V_n with respect to stronger ρ_q-metrics. This convergence is established in Theorem 3; our proof makes use of the nearly linear bounds of Theorem 1.

Let $b_n = (2 \log \log n)^{\frac{1}{2}}$ and let

$$\mathbb{B} = \{f \in C[0, 1] : f(0) = 0 = f(1), f = \textstyle\int_0^\cdot f'\, dI, \int_0^1 (f')^2\, dI \leqq 1\}.$$

Let $\mathcal{Q}$ denote the set of positive continuous functions on [0, 1] which are nondecreasing on $[0, \frac{1}{2}]$, symmetric about $\frac{1}{2}$, and have $\int_0^1 q^{-2}\, dI < \infty$. For $\delta > 0$, let $\mathcal{Q}_\delta$ denote the subset of $\mathcal{Q}$ having $\int_0^1 q^{-2-\delta}\, dI < \infty$. The functions $q(t) = [t(1-t)]^{\frac{1}{2}-\gamma}$, $\delta/2(2+\delta) < \gamma < \frac{1}{2}$, are all in $\mathcal{Q}_\delta$; the functions $q(t) = [t(1-t)]^{\frac{1}{2}}[-\log(t(1-t))]^{\frac{1}{2}+\gamma}$ with $\gamma > 0$ are in $\mathcal{Q}$ but are not in $\mathcal{Q}_\delta$ for any $\delta > 0$.

The following theorem has been proved by James [11] for a class of functions slightly larger than $\mathscr{Q}$. For $q \in \mathscr{Q}_\delta$ for some $\delta > 0$ another proof is given in [20].

THEOREM 2 (James). *If $q \in \mathscr{Q}$, then with probability one the sequence $\{U_n/b_n, n \geqq 1\}$ is relatively compact with respect to ρ_q with limit set $\mathbb{B}$.*

If f is a function on $[0, 1]$ and $n \geqq 1$ let f^* denote the function which equals f on $[1/n, 1-(1/n)]$ and equals zero elsewhere (when $n = 1$ set $f^* \equiv 0$).

THEOREM 3. *If $q \in \mathscr{Q}_\delta$ for some $\delta > 0$ then with probability one the sequence $\{V_n^*/b_n, n \geqq 1\}$ is relatively compact with respect to ρ_q with limit set $\mathbb{B}$. Moreover, if $f \in \mathbb{B}$ and $\{n'\}$ is a subsequence such that*

$$\rho_q(U_{n'}/b_{n'}, f) \to 0 \qquad as \quad n' \to \infty$$

then

$$\rho_q(V_{n'}/b_{n'}, -f) \to 0 \qquad as \quad n' \to \infty .$$

PROOF. Suppose $q \in \mathscr{Q}_\delta$. Note that

$$V_n = -U_n(\Gamma_n^{-1}) + n^{\frac{1}{2}}(\Gamma_n(\Gamma_n^{-1}) - I) . \tag{10}$$

Since $(n^{\frac{1}{2}}/b_n)\rho_q(\Gamma_n(\Gamma_n^{-1}), I) = \{n^{\frac{1}{2}}q(1/n)b_n\}^{-1} \to 0$ as $n \to \infty$ it suffices to show that with probability one $-U_n(\Gamma_n^{-1})^*/b_n$ is relatively compact with respect to ρ_q with limit set $\mathbb{B}$. Let $S \equiv \{f \in D[0, 1] : \rho_q(f, 0) < \infty\}$. For $f \in S$ define $f_n \equiv (b_n/n^{\frac{1}{2}})f + I$ and $f_n^-(x) \equiv \inf\{t : f_n(t) \geqq x\}$, $0 \leqq x \leqq 1$, $f_n^-(x) \equiv 1$ if the set is empty. Now define functions R_n, $n \geqq 1$, and R from (S, ρ_q) to (S, ρ_q) by

$$R_n(f) = f(f_n^-)^* , \qquad R(f) = f .$$

The fact that $f(f_n^-)^*$ is zero on $[0, 1/n)$ and $(1-(1/n), 1]$ implies that $\rho_q(f(f_n^-)^*, 0) < \infty$ for all $f \in S$ and hence R_n is well defined. Note that $R_n(U_n/b_n) = U_n(\Gamma_n^{-1})^*/b_n$. Also the ρ-compactness of $\mathbb{B}$ in S and the inequality $|f(t)| \leqq [t(1-t)]^{\frac{1}{2}}$ for all $f \in \mathbb{B}$ ([7], Lemma 1) imply that $\mathbb{B}$ is ρ_q-compact in S. (If (f_n) is a sequence in $\mathbb{B}$, then there is a subsequence (f_{n_k}) which ρ-converges to a function $f \in \mathbb{B}$. Using $|f(t)| \leqq [t(1-t)]^{\frac{1}{2}}$ to handle the supremum over $(0, \theta]$ and $[1-\theta, 1)$ we find that (f_{n_k}) ρ_q-converges to f.) Hence by Lemma 2.1 of [21] it suffices to show that $\rho_q(U_{n'}(\Gamma_{n'}^{-1})/b_{n'}, f) \to 0$ whenever $f \in \mathbb{B}$, $n' \to \infty$, and $\rho_q(U_{n'}/b_{n'}, f) \to 0$.

But

$$\rho_q(U_n(\Gamma_n^{-1})^*/b_n, f) \leqq \rho_q(U_n/b_n, f) + \rho_q(f(\Gamma_n^{-1})^*, f)$$

and therefore, by Theorem 2, it remains only to show that $\rho_q(f(\Gamma_n^{-1})^*, f) \to 0$ a.s. as $n \to \infty$. Now $\rho(\Gamma_n^{-1}, I) \to 0$ a.s. and f is continuous so $\rho(f(\Gamma_n^{-1}), f) \to 0$ a.s.; thus it suffices to show that with probability one

$$\lim_{\theta \to 0} \limsup_{n \to \infty} \sup_{0 < t \leqq \theta} |f(\Gamma_n^{-1}(t))^*|/q(t) = 0 .$$

Let $\varepsilon > 0$; choose $\tau_1 = \tau$ in Theorem 1 so that $\tau < 1 + \delta/2$ where $\int_0^1 q^{-2-\delta}\, dI < \infty$. Note that the finiteness of this integral implies that $\theta^{1/(2+\delta)}/q(\theta) \to 0$ as $\theta \to 0$. Choose θ so small that $\beta^{-(1/2\tau)}\theta^{1/(2+\delta)}/q(\theta) < \varepsilon$ and fix

ω in the set A of Theorem 1. Then for $n \geqq N(\omega, \tau)$,

$$\begin{aligned}\sup_{0<t\leqq\theta} |f(\Gamma_n^{-1}(t))^*|/q(t) &\leqq \sup_{1/n<t\leqq\theta} (\Gamma_n^{-1}(t))^{\frac{1}{2}}/q(t)\\ &\leqq \sup_{0<t\leqq\theta} (t/\beta)^{1/2\tau}/q(t)\\ &\leqq \beta^{-(1/2\tau)} \sup_{0<t\leqq\theta} t^{1/(2+\delta)}/q(t)\\ &< \varepsilon\end{aligned}$$

where the second inequality follows from (8) of Theorem 1. □

4. A law of the iterated logarithm for linear combinations of order statistics. Now we are prepared to use Theorem 2 in conjunction with Theorems 1 and 3 to establish a law of the iterated logarithm for the statistic T_n defined in (1). Theorem 4 here parallels Theorem 1 of [16]; a theorem parallel to Theorem 2 of [16] may be proved using our Theorem 3. Throughout this and the following section $b_n = (2 \log \log n)^{\frac{1}{2}}$.

Set

$$\mu_n = \int_0^1 g_n J_n \, dI + \sum_1^\kappa d_k g_n(p_k)$$

where $J_n(t)$ equals c_{ni} for $(i-1)/n < t \leqq i/n$, $1 \leqq i \leqq n$, $J_n(0) = c_{n1}$, and I is Lebesgue measure. Here $d_1, \cdots, d_\kappa$ are finite nonzero constants.

For fixed b_1, b_2 and $M > 0$ define a "scores bounding function" B by

$$B(t) = Mt^{-b_1}(1-t)^{-b_2}, \qquad 0 < t < 1.$$

For fixed $\delta > 0$ define

$$D(t) = Mt^{-\frac{1}{2}+b_1+\delta}(1-t)^{-\frac{1}{2}+b_2+\delta}, \qquad 0 < t < 1,$$

and

$$q(t) = [t(1-t)]^{\frac{1}{2}-\delta/2}, \qquad 0 < t < 1.$$

Let g denote a fixed function in $\mathscr{G}$ and let J denote a fixed measurable function on (0, 1).

ASSUMPTION 1 (Boundedness). Let $|g| \leqq D$, all $|g_n| \leqq D$, $|J| \leqq B$, all $|J_n| \leqq B$ on (0, 1), and suppose $\int_0^1 Bq \, d|g| < \infty$.

ASSUMPTION 2 (Smoothness). Except on a set of t's of $|g|$-measure 0 we have both J is continuous at t and $J_n \to J$ uniformly in some small neighborhood of t as $n \to \infty$.

ASSUMPTION 3 (Convergence). $\int_0^1 Bq \, d|g_n - g| \to 0$ as $n \to \infty$.

ASSUMPTION 4 ($\kappa > 0$). For $1 \leqq k \leqq \kappa$ we have $n^{\frac{1}{2}}(d_{nk} - d_k)/b_n \to 0$ as $n \to \infty$. In some small neighborhood of each of $p_1, \cdots, p_\kappa$ the functions g_n' for $n \geqq 1$ form an equiuniformly continuous family and $g_n(p_k) \to g(p_k)$ and $g_n'(p_k) \to g'(p_k)$ as $n \to \infty$ for each $1 \leqq k \leqq \kappa$. (If $g_n = g$ for all n we require at each of $p_1, \cdots, p_\kappa$ only that $g'(p_k)$ exist.)

Define

$$\begin{aligned}\sigma^2 = &\int_0^1 \int_0^1 (s \wedge t - st) J(s) J(t) \, dg(s) \, dg(t)\\ &+ 2 \sum_1^\kappa d_k g'(p_k) \int_0^1 (t \wedge p_k - t_{p_k}) J(t) \, dg(t)\\ &+ \sum_1^\kappa \sum_1^\kappa d_j d_k g'(p_j) g'(p_k)(p_j \wedge p_k - p_j p_k).\end{aligned}$$

THEOREM 4. *If Assumptions* 1, 2, 3 *and* 4 *hold*, *then with probability one the sequence*

$$\{n^{\frac{1}{2}}(T_n - \mu_n)/b_n, n \geqq 1\} \tag{11}$$

is relatively compact with limit set

$$\begin{aligned} \mathbb{L} &\equiv \{-\textstyle\int_0^1 Jf\,dg - \sum_1^\kappa d_k g'(p_k) f(p_k) : f \in \mathbb{B}\} \\ &= [-\sigma, +\sigma] . \end{aligned} \tag{12}$$

Moreover, $\sup \mathbb{L} = \sigma$ *is achieved at the function* f_M *in* $\mathbb{B}$ *with*

$$f_M(t) = -\frac{1}{\sigma}\{\textstyle\int_0^1 (s \wedge t - st) J(s)\,dg(s) + \sum_1^\kappa d_j g'(p_j)(p_j \wedge t - p_j t)\} .$$

PROOF. Consider the case $\kappa = 0$; we will later extend to $\kappa > 0$ by use of Theorem 3. From [16]

$$n^{\frac{1}{2}}(T_n - \mu_n) = -(S_n + \gamma_{n1} + \gamma_{n2} + \gamma_{n3}) \quad \text{a.s.}$$

where

$$\begin{aligned} S_n &\equiv \textstyle\int_{\xi_{n1}}^{\xi_{nn}} A_n U_n\,dg_n = \int_0^1 A_n^* U_n\,dg_n , \\ \gamma_{n1} &\equiv n^{\frac{1}{2}} g_n(\xi_{n1})(\Psi_n(0) - \Psi_n(\xi_{n1})) , \\ \gamma_{n2} &\equiv n^{\frac{1}{2}} g_n(\xi_{nn}) \Psi_n(\xi_{nn}) , \\ \gamma_{n3} &\equiv n^{\frac{1}{2}} \textstyle\int_{[\xi_{n1}, \xi_{nn})^c} g_n J_n\,dI , \\ \Psi_n(t) &\equiv -\textstyle\int_t^1 J_n\,dI , \\ A_n &\equiv [\Psi_n(\Gamma_n) - \Psi_n(I)]/(\Gamma_n - I) , \end{aligned}$$

and where we now set A_n^* equal to A_n for $t \in [\xi_{n1}, \xi_{nn}]$ and zero otherwise. Let $q'(t) \equiv q(t)[t(1-t)]^{\delta/4} \leqq q(t)$.

We first show that S_n/b_n is relatively compact with limit set $-\mathbb{L}$. Define functions R_n, $n \geqq 1$, and R from $(S, \rho_{q'})$ (with S defined as in the proof of Theorem 3) to $(\mathbb{R}, |\ |)$ by

$$R_n(f) = \textstyle\int_{a_{n1}}^{a_{nn}} A_n(f) f\,dg_n , \qquad R(f) = \int_0^1 Jf\,dg$$

where $A_n(f)$ is A_n with Γ_n replaced by $f_n \equiv (b_n/n^{\frac{1}{2}})f + I$, $a_{n1} \equiv f_n^-(1/n)$, $a_{nn} \equiv f_n^-(1 - 1/n)$, and $f_n^-(x) \equiv \inf\{t : f_n(t) \geqq x\}$ as in Section 3. With this definition of R_n, $R_n(U_n/b_n) = S_n/b_n$, and R_n is well defined, at least for large n, for all f in S. Note that with $\kappa = 0$, $R(\mathbb{B}) = -\mathbb{L}$. Now, by the continuous mapping Lemma 2.1 of [21], it suffices to show that $S_{n_j}/b_{n_j} \to \int_0^1 Jf\,dg = R(f)$ whenever $f \in \mathbb{B}$, $n_j \to \infty$, and $\rho_{q'}(U_{n_j}/b_{n_j}, f) \to 0$. We have

$$\begin{aligned} |S_n/b_n - R(f)| &\leqq \gamma_{n4} + \gamma_{n5} + \gamma_{n6} \\ &\equiv \textstyle\int_0^1 |A_n^* U_n/b_n|\,d|g_n - g| + \int_0^1 |A_n^* - J|\,|U_n/b_n|\,d|g| \\ &\quad + \textstyle\int_0^1 |J|\,|U_n/b_n - f|\,d|g| . \end{aligned}$$

Also, by Assumption 1, when $b_1, b_2 > 0$,

$$\begin{aligned} |A_n| &\leqq |\textstyle\int_I^{\Gamma_n} J_n\,dI/(\Gamma_n - I)| \\ &\leqq \textstyle\int_I^{\Gamma_n} B\,dI/(\Gamma_n - I) \leqq B \vee B(\Gamma_n) . \end{aligned}$$

Now Theorem 1 comes into play. Fix $\omega \in A$ of Theorem 1 and choose τ_1, τ_2 there so that $b_1\tau_1 = b_1 + \delta/4$ and $b_2\tau_2 = b_2 + \delta/4$. Then, with $n \geqq N_\omega$ of Theorem 1, (5) and (6) imply

$$|A_n^*| \leqq M_{1,2} M I^{-b_1\tau_1}(1-I)^{-b_2\tau_2} = M_{1,2} B[I(1-I)]^{-\delta/4} \tag{13}$$

for some constant $M_{1,2}$ depending on β of Theorem 1. Clearly, (13) also holds if either b_1 or b_2 equals zero. In the case b_1 or $b_2 < 0$, use (4) of Theorem 1 in place of (5) and (7) and an argument similar to that given for $b_1, b_2 > 0$. Using (13), Theorem 2, and Assumption 3

$$\gamma_{n4} \leqq \rho_{q'}(U_n/b_n, 0) M_{1,2} \int_0^1 Bq\, d|g_n - g| \to 0 \qquad \text{a.s.} \quad \text{as} \quad n \to \infty .$$

To handle γ_{n5} write

$$\gamma_{n5} \leqq \rho_{q'}(U_n/b_n, 0) \int_0^1 |A_n^* - J| q'\, d|g| .$$

Use (13) again to find $|A_n^* - J|q'$ dominated by the $|g|$-integrable function $(M_{1,2} + 1)Bq$. Since $A_n(t) \to J(t)$ a.e. $|g|$ by Assumption 2, Theorem 2 and dominated convergence show that $\gamma_{n5} \to 0$ a.s. Now

$$\gamma_{n6} \leqq \rho_{q'}(U_n/b_n, f) \int_0^1 Bq\, d|g| ,$$

and hence $\gamma_{n6} \to 0$ a.s. whenever $n_j \to \infty$ and $\rho_{q'}(U_{n_j}/b_{n_j}, f) \to 0$.

Thus S_n/b_n is a.s. relatively compact with limit set $-\mathbb{L}$. To complete the proof in the case $\kappa = 0$, it remains only to show that $\sum_1^3 \gamma_{ni}/b_n \to 0$ a.s. From [16] it suffices to show that

$$Y_n \equiv n^{\frac{1}{2}} \xi_{n1}^{\frac{1}{2}+\delta}/b_n \to 0 \qquad \text{a.s.} \quad n \to \infty .$$

But with

$$a_n \equiv (n^{\frac{1}{2}}/b_n)(2 \log n/n)^{\frac{1}{2}+\delta} \to 0 \qquad \text{as} \quad n \to \infty ,$$

and $A_n \equiv \{Y_n \geqq a\}$, $P(A_n \text{ i.o.}) = 0$ since

$$\begin{aligned} P(A_n) &= P(\xi_{n1} \geqq (b_n a_n/n^{\frac{1}{2}})^{1/(\frac{1}{2}+\delta)}) \\ &= (1 - 2 \log n/n)^n \\ &\leqq n^{-2} \end{aligned}$$

which converges when summed on n. Hence $Y_n \to 0$ a.s. and $\sum_1^3 \gamma_{ni}/b_n \to 0$ a.s. as $n \to \infty$, completing the proof in the case $\kappa = 0$.

Now suppose $\kappa > 0$. Without loss set $\kappa = 1$, $d_{n1} = d_n$, $d_1 = d$, and $p_1 = p$. In addition to the term considered above, (11) contributes

$$F_n \equiv n^{\frac{1}{2}}[d_n g_n(\xi_{n,[np]+1}) - d g_n(p)]/b_n ;$$

it remains only to show that F_n is relatively compact with limit set $\mathbb{F} = \{-dg'(p)f(p), f \in \mathbb{B}\}$. This follows easily from Assumption 4 and Theorem 3.

To complete the proof we must show that $\sup \mathbb{L} = \sigma$ and that the function f in $\mathbb{B}$ attaining the supremum is f_M, as claimed. To evaluate

$$\sup \mathbb{L} = \sup\{-\textstyle\int_0^1 Jf\, dg - \sum_1^\kappa d_k g'(p_k) f(p_k) : f \in \mathbb{B}\} ,$$

break the integral in the first term at $\frac{1}{2}$ (assuming that $\frac{1}{2}$ is a continuity point of g; if not, break the integration at a continuity point in a neighborhood of $\frac{1}{2}$) and write

$$\begin{aligned} -\int_0^1 Jf\,dg &= -\int_0^{\frac{1}{2}} J(t) \int_0^t f'(s)\,ds\,dg(t) + \int_{\frac{1}{2}}^1 J(t) \int_t^1 f'(s)\,ds\,dg(t) \\ &= -\int_0^{\frac{1}{2}} \int_s^{\frac{1}{2}} J(t)f'(s)\,dg(t)\,ds + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^s J(t)f'(s)\,dg(t)\,ds \\ &= \int_0^1 f'(s)P_1(s)\,ds \end{aligned}$$

where

$$\begin{aligned} P_1(s) &\equiv -\int_s^{\frac{1}{2}} J(t)\,dg(t) \quad 0 \leqq s < \tfrac{1}{2} \\ &\equiv +\int_{\frac{1}{2}}^s J(t)\,dg(t) \quad \tfrac{1}{2} \leqq s \leqq 1\,. \end{aligned}$$

Here the interchanges of order of integration are justified by Fubini's theorem since, using $\int_0^1 (f')^2\,dI \leqq 1$ and Cauchy–Schwarz,

$$\begin{aligned} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} |J(t)I_{[0,t]}(s)f'(s)|\,ds\,d|g|(t) &\leqq \int_0^{\frac{1}{2}} |J(t)|t^{\frac{1}{2}}\,d|g|(t) \\ &\leqq 2\int_0^{\frac{1}{2}} Bq\,d|g| < \infty\,. \end{aligned}$$

In the same way, but with less difficulty (now there is no problem in interchanging the summation and integration and no need to work with two regions), rewrite the second term as

$$-\sum_1^\kappa d_k g'(p_k)f(p_k) = \int_0^1 f'(s)P_2(s)\,ds$$

where

$$P_2(s) \equiv -\sum_1^\kappa d_k g'(p_k)I_{[0,p_k]}(s)\,.$$

Therefore, with $P \equiv P_1 + P_2$ and $L(f)$ denoting an element of the set $\mathbb{L}$, for all f in $\mathbb{B}$,

$$L(f) = \int_0^1 f'P\,dI\,.$$

Also, all f in $\mathbb{B}$ satisfy $\int_0^1 f'\,dI = 0$, and hence, for any real α and all $f \in \mathbb{B}$,

$$\begin{aligned} |L(f)| &= |\int_0^1 f'(P-\alpha)\,dI| \\ &\leqq (\int_0^1 (f')^2\,dI \int_0^1 (P-\alpha)^2\,dI)^{\frac{1}{2}} \\ &\leqq (\int_0^1 (P-\alpha)^2\,dI)^{\frac{1}{2}} \end{aligned}$$

where the first inequality is Cauchy–Schwarz and the second inequality follows from $\int_0^1 (f')^2\,dI \leqq 1$ for all f in $\mathbb{B}$. Equality holds in the first case if $f' = \beta(P-\alpha)$ for some constant β, and equality holds in the second case if $\int_0^1 (f')^2\,dI = 1$. Define $f_M'(s) \equiv \beta(P(s)-\alpha)$ and $f_M(t) \equiv \int_0^t f_M'(s)\,ds$. We want $f_M \in \mathbb{B}$, so choose α to make $f_M(1) = 0$:

$$\begin{aligned} 0 &= f_M(1) \\ &= \beta(-\int_{[0,\frac{1}{2}]} \int_{[s,\frac{1}{2}]} J(t)\,dg(t)\,ds + \int_{(\frac{1}{2},1]} \int_{[\frac{1}{2},s)} J(t)\,dg(t)\,ds - \sum_1^\kappa d_k g'(p_k)p_k - \alpha)\,. \end{aligned}$$

Interchanging the order of integration and solving for α yields

$$\alpha = -\int_{[0,\frac{1}{2}]} tJ(t)\,dg(t) + \int_{[\frac{1}{2},1]} (1-t)J(t)\,dg(t) - \sum_1^\kappa d_k g'(p_k)p_k\,.$$

Therefore, for $0 \leqq s \leqq 1$,

$$P(s) - \alpha = -\int_0^1 (I_{[s,1]}(t) - t)J(t)\,dg(t) - \sum_1^\kappa d_k g'(p_k)(I_{[0,p_k]}(s) - p_k)$$

and hence

$$\begin{aligned}\int_0^1 (f_M')^2\,dI &= \beta^2 \int_0^1 (P(r) - \alpha)^2\,dr \\ &= \beta^2(\int_0^1 \int_0^1 \int_0^1 (I_{[r,1]}(s) - s)(I_{[r,1]}(t) - t)J(s)J(t)\,dg(s)\,dg(t)\,dr \\ &\qquad + 2\int_0^1 \int_0^1 (I_{[r,1]}(t) - t)J(t)\,dg(t) \sum_1^\kappa d_k g'(p_k)(I_{[0,p_k]}(r) - p_k)\,dr \\ &\qquad + \int_0^1 \sum_1^\kappa \sum_1^\kappa d_j d_k g'(p_j)g'(p_k)(I_{[0,p_j]}(r) - p_j)(I_{[0,p_k]}(r) - p_k)\,dr \\ &= \beta^2(\int_0^1 \int_0^1 (s \wedge t - st)J(s)J(t)\,dg(s)\,dg(t) \\ &\qquad + 2\sum_1^\kappa d_k g'(p_k) \int_0^1 (t \wedge p_k - tp_k)J(t)\,dg(t) \\ &\qquad + \sum_1^\kappa \sum_1^\kappa d_j d_k g'(p_j)g'(p_k)(p_j \wedge p_k - p_j p_k)) \\ &= \beta^2\rho^2\end{aligned}$$

since

$$\int_0^1 (I_{[r,1]}(s) - s)(I_{[r,1]}(t) - t)\,dr = s \wedge t - st\,,$$

$$\int_0^1 (I_{[r,1]}(t) - t)(I_{[0,p_k]}(r) - p_k)\,dr = t \wedge p_k - tp_k\,,$$

and

$$\int_0^1 (I_{[0,p_j]}(r) - p_j)(I_{[0,pk]}(r) - p_k)\,dr = p_j \wedge p_k - p_j p_k\,.$$

Hence, by defining $\beta = 1/\sigma$, we have $f_M' = \sigma^{-1}(P - \alpha)$, $f_M(1) = 0$, $\int_0^1 (f_M')^2\,dI = 1$, and $\sup \mathbb{L} = \sigma$. □

5. Examples. The following examples parallel those of [16].

EXAMPLE 1. Let $X_1, \cdots X_n$ be a random sample from an arbitrary df F for which $E|X|^r < \infty$ for some $r > 0$. Let

$$T_n = n^{-1} \sum_1^n J(t_{ni})X_{ni}$$

where $\max_{1 \leqq i \leqq n} |t_{ni} - i/n| \to 0$ as $n \to \infty$ and where for some $a > 0$

$$a\left[\frac{i}{n} \wedge \left(1 - \frac{i}{n}\right)\right] \leqq t_{ni} \leqq 1 - a\left[\frac{i}{n} \wedge \left(1 - \frac{i}{n}\right)\right], \qquad 1 \leqq i \leqq n\,.$$

Suppose J is continuous except at a finite number of points at which F^{-1} is continuous, and suppose

$$|J(t)| \leqq M[t(1 - t)]^{-\frac{1}{2}+1/r+\delta}\,, \qquad 0 < t < 1$$

for some $\delta > 0$. Then, with probability one, the sequence

$$\{n^{\frac{1}{2}}(T_n - \int_0^1 J_n F^{-1}\,dI)/b_n,\ n \geqq 1\}$$

is relatively compact with limit set

$$\{-\int_0^1 Jf\,dF^{-1}, f \in \mathbb{B}\} = [-\sigma, +\sigma]$$

where

$$\sigma^2 = \int_0^1 \int_0^1 (s \wedge t - st)J(s)J(t)\,dF^{-1}(s)\,dF^{-1}(t)\,.$$

PROOF. Analogous to the proof of Example 1 of [16]. □

EXAMPLE 1a. Let $X_1, \cdots X_n$ be a random sample from a df having $E|X|^r < \infty$ for some $r > 2$ and let $\sigma^2 = \text{Var}(X)$. Then with probability one the sequence

$$\{n^{\frac{1}{2}}(\bar{X} - E(X))/b_n, n \geqq 1\}$$

is relatively compact with limit set

$$\{-\textstyle\int_0^1 f\, dF^{-1}, f \in \mathbb{B}\} = [-\sigma, +\sigma]\,.$$

This example shows that the Hartmann–Wintner law of the iterated logarithm [9] "just fails" to be a corollary to Theorem 4.

EXAMPLE 1b. Let $X_1, \cdots, X_n$ be a random sample from the $N(0, 1)$ df Φ. For integral $r > 0$ let

$$T_n = n^{-1} \textstyle\sum_1^n [\Phi^{-1}(i/(n+1))]^r X_{ni}$$

and let $\sigma_r^2 = \text{Var}(X^{r+1})$. Then, with probability one the sequence

$$\{n^{\frac{1}{2}}(T_n - E(X^{r+1}))/b_n, n \geqq 1\}$$

is relatively compact with limit set

$$\{-\textstyle\int_0^1 [\Phi^{-1}]^r f\, d\Phi^{-1}, f \in \mathbb{B}\} = [-\sigma_r, +\sigma_r]\,.$$

EXAMPLE 2 (The linearly trimmed mean). Let $0 < a < \frac{1}{2}$ be fixed, and let $a_n = [na]$. Let $X_1, \cdots, X_n$ be a random sample from $F_\theta = F(\cdot - \theta)$ where F is any df symmetric about 0. For n even, define

$$T_n = \textstyle\sum_{a_n+1}^{n/2} [2(i - a_n) - 1](X_{ni} + X_{n,n-i+1}) \Big/ 2\left(\frac{n}{2} - a_n\right)^2.$$

Then with probability one the sequence

$$\{n^{\frac{1}{2}}(T_n - \theta)/b_n, n \geqq 1\}$$

is relatively compact with limit set

$$\{-\textstyle\int_0^1 Jf\, dF^{-1}, f \in \mathbb{B}\}$$

where $J(t) = 0$, $0 \leqq t \leqq a$, $J(t) = 4(t - a)/(1 - 2a^2)$, $a \leqq t \leqq \frac{1}{2}$, and $J(t) = J(1 - t)$, $\frac{1}{2} \leqq t \leqq 1$.

EXAMPLE 3. Let $X_1, \cdots, X_n$ be independent Bernoulli (θ) rv's. Let $g = F^{-1}$. Thus $g(t) = -\infty, 0, 1$ for $t = 0, 0 < t \leqq 1 - \theta, 1 - \theta < t \leqq 1$. Let $J(t)$ equal 0, 1 for $0 \leqq t < \frac{1}{2}, \frac{1}{2} \leqq t \leqq 1$ and let $c_{ni} = J(i/n)$. Then T_n equals $\frac{1}{2}$ if more than half of the X_i's are positive; while T_n equals the proportion of positive X_i's if less than half of the X_i's are positive.

(a) Suppose $\theta = \frac{1}{2}$. Then $n^{\frac{1}{2}}(T_n - \mu_n)/b_n = n^{\frac{1}{2}}(T_n - \frac{1}{2})/b_n$ has lim sup 0 and lim inf $-\frac{1}{2}$ with probability one. J is not continuous a.e. $|g|$ and hence the hypotheses of Theorem 4 fail.

(b) Suppose $\theta > \frac{1}{2}$. Then $n^{\frac{1}{2}}(T_n - \mu_n)/b_n = n^{\frac{1}{2}}(T_n - \frac{1}{2})/b_n$ has both lim sup and lim inf 0 with probability one by Theorem 4.

(c) Suppose $\theta < \frac{1}{2}$ and set $\sigma_\theta^2 = \theta(1 - \theta)$. Then with probability one the sequence $n^{\frac{1}{2}}(T_n - \mu_n)/b_n = n^{\frac{1}{2}}(T_n - \theta)/b_n$ has limit set $[-\sigma_\theta, +\sigma_\theta]$ by Theorem 4.

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