Testing for sparse normal means: an update

Jon A. Wellner

University of Washington, visiting Heidelberg
• joint work with Leah Jager, U. S. Naval Academy
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• Email: jaw@stat.washington.edu
  http://www.stat.washington.edu/jaw/jaw.research.html
Outline

• Testing problems for normal means
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- Detection boundaries and Tukey’s “higher criticism” statistic
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- Beyond normality: generalized Gaussian distributions and...
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• Beyond normality: generalized Gaussian distributions and ...
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• Problems and Questions
1. Testing problems for sparse normal means

- Initial setting: multiple testing of normal means
  For $i = 1, \ldots, n$ consider testing

  \[ H_{0,i} : X_i \sim N(0, 1) \]

  versus

  \[ H_{1,i} : X_i \sim N(\mu_i, 1) \text{ with } \mu_i > 0. \]
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• Sparsity: proportion $\epsilon_n \equiv n^{-1} \# \{ i \leq n : \mu_i > 0 \}$ is small;
  $\epsilon_n \sim n^{-\beta}$ with $0 < \beta < 1$. 
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  - Q3: Which null hypotheses are false?

- Main focus here: Q1.
• Previous work: Q1: is there any signal?
  ◦ Ingster (1997, 1999)
  ◦ Jin (2004)
  ◦ Donoho and Jin (2004)
  ◦ Jager and Wellner (2007)
  ◦ Hall and Jin (2007)
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• Previous work: Q2: What is the proportion of non-null hypotheses?
  ◦ Swanepoel (1999)
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• Previous work: Q3: Where is the signal and how big is it?
  ◦ Benjamini and Hochberg (1995)
  ◦ Efron, Tibshirani, Storey, and Tusher (2001)
  ◦ Storey, Dai, and Leek (2005)
  ◦ Donoho and Jin (2006)
2. Detection boundaries and Tukey’s “higher criticism statistic

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• Suppose $Y_1, \ldots, Y_n$ i.i.d. $G$ on $\mathbb{R}$
• test $H : G = N(0, 1)$ versus
  $H_1 : G = (1 - \epsilon)N(0, 1) + \epsilon N(\mu, 1)$, and, in particular, against

  $$H_1^{(n)} : G = (1 - \epsilon_n)N(0, 1) + \epsilon_n N(\mu_n, 1).$$

  for $\epsilon_n = n^{-\beta}$, $\mu_n = \sqrt{2r \log n}$
  $0 < \beta < 1$, $0 < r < 1$. 
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- Let $\Phi(z) \equiv P(Z \leq z) = \int_{-\infty}^{z} (2\pi)^{-1/2} \exp(-x^2/2)dx$, $Z \sim N(0, 1)$.
• transform to $X_i \equiv 1 - \Phi(Y_i) \in [0, 1]$ i.i.d.

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• Then the testing problem becomes: test

$$H_0 : F = F_0 = U(0, 1) \quad \text{versus}$$

$$H_1^{(n)} : F(u) = u + \epsilon_n \left\{ (1 - u) - \Phi(\Phi^{-1}(1 - u) - \mu_n) \right\}$$

$$= (1 - \epsilon_n)u + \epsilon_n \left\{ 1 - \Phi(\Phi^{-1}(1 - u) - \mu_n) \right\}$$
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$$= (1 - \epsilon_n)u + \epsilon_n \{1 - \Phi(\Phi^{-1}(1 - u) - \mu_n)\}$$

• Test statistics: Donoho-Jin

$$HC^*_n \equiv \sup_{X_{(1)} \leq u < X_{[n/2]}} \frac{\sqrt{n(\overline{F}_n(u) - u)}}{\sqrt{u(1 - u)}}$$

$$\equiv \text{Tukey’s “higher criticism statistic”}$$

where $\overline{F}_n(u) \equiv n^{-1} \sum_{i=1}^{n} 1_{[0,u]}(X_i) = \text{empirical distribution function of the } X_i \text{’s.}$
• Optimal detection boundary $\rho^*(\beta)$ defined by:

$$
\rho^*(\beta) = \begin{cases} 
\beta - 1/2, & 1/2 < \beta \leq 3/4 \\
(1 - \sqrt{1 - \beta})^2, & 3/4 < \beta < 1
\end{cases}
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• **Theorem 1**: (Donoho - Jin, 2004). For $r > \rho^*(\beta)$ the test based on $HC^*_n$ is size and power consistent for testing $H_0$ versus $H_1^{(n)}$. 
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• **Theorem 1**: (Donoho - Jin, 2004). For $r > \rho^*(\beta)$ the test based on $HC_n^*$ is size and power consistent for testing $H_0$ versus $H_1^{(n)}$.

• With $h_n(\alpha_n) = \sqrt{2\log\log(n)(1 + o(1))}$

$$P_{H_0}(HC_n^* > h_n(\alpha_n)) = \alpha_n \to 0, \quad \text{and}$$

$$P_{H_1^{(n)}}(HC_n^* > h_n(\alpha_n)) \to 1, \quad \text{as } n \to \infty.$$
Figure 1. Detection boundary: \( r > \rho^*(\beta) \) detectable
Some alternative statistics:

- Berk-Jones (1979) test statistic:

\[ R_n \equiv \sup \log \lambda_n(x) = \sup \log K(F_n(x), F_0(x)) \]

with

\[ K(u, v) \equiv u \log \left( \frac{u}{v} \right) + (1-u) \log \left( \frac{1-u}{1-v} \right) \]
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\]

\[
K(u, v) \equiv u \log \left( \frac{u}{v} \right) + (1 - u) \log \left( \frac{1 - u}{1 - v} \right)
\]

- Adaptation to one-sided \( p \)-value setting:

\[
BJ_n^+ \equiv n \sup_{X(1) \leq u \leq 1/2} K^+(F_n(u), u)
\]

where

\[
K^+(u, v) \equiv \begin{cases} 
K(u, v), & \text{if } 0 < v < u < 1, \\
0, & \text{if } 0 \leq u \leq v \leq 1, \\
+\infty, & \text{otherwise}.
\end{cases}
\]
• **Theorem 2:** (Donoho - Jin, 2004). For \( r > \rho^*(\beta) \) the test based on \( BJ_n^+ \) is size and power consistent for testing \( H_0 \) versus \( H_1^{(n)} \); i.e. with \( h_n(\alpha_n) = \sqrt{2 \log \log(n)}(1 + o(1)) \)

\[
P_{H_0}(BJ_n^+ > h_n(\alpha_n)) = \alpha_n \to 0, \quad \text{and} \\
P_{H_1^{(n)}}(BJ_n^+ > h_n(\alpha_n)) \to 1, \quad \text{as } n \to \infty.
\]
3. A new family of statistics via phi-divergences

A family of test statistics connecting “Higher criticism” and Berk-Jones:

• For $s \in \mathbb{R}, x \geq 0$ define

$$
\phi_s(x) = \begin{cases} 
\frac{1-s+sx-x^s}{s(1-s)}, & s \neq 0, 1 \\
x \log x - x + 1, & s = 1 \\
-\log x + x - 1, & s = 0.
\end{cases}
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- Then define

$$K_s(u, v) = v\phi_s(u/v) + (1-v)\phi_s((1-u)/(1-v)).$$
• Special cases:

\[ K_1(u, v) = K(u, v) = u \log(u/v) + (1 - u) \log((1 - u)/(1 - v)) \]
\[ K_0(u, v) = K(v, u) \]
\[ K_2(u, v) = \frac{1}{2} \frac{(u - v)^2}{v(1 - v)} \]
\[ K_{-1}(u, v) = K_2(v, u) = \frac{1}{2} \frac{(u - v)^2}{u(1 - u)} \]
\[ K_{1/2}(u, v) = 2\{(\sqrt{u} - \sqrt{v})^2 + (\sqrt{1 - u} - \sqrt{1 - v})^2\} = 4\{1 - \sqrt{uv} - \sqrt{(1 - u)(1 - v)}\}. \]
The new family of statistics:

\[ S_n(s) = \begin{cases} 
\sup_{x \in \mathbb{R}} K_s(F_n(x), F_0(x)), & s \geq 1 \\
\sup_{x \in [X_{(1)}, X_{(n)}]} K_s(F_n(x), F_0(x)), & s < 1,
\end{cases} \]
• The new family of statistics:

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\sup_{x \in [X(1), X(n)]} K_s(F_n(x), F_0(x)), & s < 1, 
\end{cases} \]

• Thus, with \( F_0(x) = x \),

\[
\begin{align*}
S_n(1) &= R_n, & S_n(0) &= \text{“reversed” Berk-Jones} \equiv \tilde{R}_n \\
S_n(2) &= \frac{1}{2} \sup_{x \in \mathbb{R}} \frac{(F_n(x) - x)^2}{x(1 - x)}, \\
S_n(-1) &= \frac{1}{2} \sup_{x \in [X(1), X(n)]} \frac{(F_n(x) - x)^2}{F_n(x)(1 - F_n(x))}, \\
S_n(1/2) &= 4 \sup_{x \in [X(1), X(n)]} \left\{ 1 - \sqrt{F_n(x)x} - \sqrt{(1 - F_n(x))(1 - x)} \right\}
\end{align*}
\]
• Version of the statistics for one-sided $p$-value setting:

$$S_n^+ \equiv n \sup_{X(1) \leq u \leq 1/2} K_s^+ (F_n(u), u)$$

where

$$K_s^+ (u, v) \equiv \begin{cases} 
K_s(u, v), & \text{if } 0 < v < u < 1, \\
0, & \text{if } 0 \leq u \leq v \leq 1, \\
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+\infty, & \text{otherwise.}
\end{cases} \]

• Theorem: (Jager - Wellner, 2007). For \( r > \rho^*(\beta) \) the tests based on \( S_n^+ (s) \) with \(-1 \leq s \leq 2\) are size and power consistent for testing \( H_0 \) versus \( H_1^{(n)} \); i.e. With \( s_n(\alpha_n) = \log \log (n)(1 + o(1)) \)

\[ P_{H_0} (S_n^+ > s_n(\alpha_n)) = \alpha_n \to 0, \quad \text{and} \]
\[ P_{H_1^{(n)}} (S_n^+ > s_n(\alpha_n)) \to 1, \quad \text{as } n \to \infty. \]
Figure 2. Separation plots: \( n = 5 \times 10^5, r = .15, \beta = 1/2 \)

Smoothed histograms of reps = 200 of the statistics under the null hypothesis and the the alternative hypothesis
4. Beyond normality:

generalized Gaussian distributions, ...

- Donoho and Jin (2004) also computed detection boundaries for sparse mixtures of “Generalized Gaussian” or Subbotin distributions: $X \sim GN_{\gamma}(\mu)$ has density function

$$f_{\gamma,\mu}(x) = \frac{1}{C_{\gamma}} \exp\left(-\frac{|x - \mu|^{\gamma}}{\gamma}\right), \quad C_{\gamma} = 2\Gamma(1/\gamma)\gamma^{1/\gamma-1}.$$
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$$f_{\gamma,\mu}(x) = \frac{1}{C_{\gamma}} \exp \left( -\frac{|x - \mu|^{\gamma}}{\gamma} \right), \quad C_{\gamma} = 2\Gamma(1/\gamma)\gamma^{1/\gamma - 1}.$$

- Suppose $Y_1, \ldots, Y_n$ i.i.d. $G$ on $\mathbb{R}$.

- Test $H_0 : G = GN_{\gamma}(0)$ versus

$$H_1^{(n)} : G = (1 - \epsilon_n)GN_{\gamma}(0) + \epsilon_nGN_{\gamma}(\mu_n) \text{ where}$$

$$\epsilon_n = n^{-\beta}, \quad \mu_{\gamma,n} = (\gamma r \log n)^{1/\gamma},$$

where $1/2 < \beta < 1, \quad 0 < r < 1$. 
 Detection boundary for $1 < \gamma \leq 2$:

$$
\rho^*_\gamma(\beta) = \begin{cases} 
(2^{1/(\gamma-1)} - 1)\gamma^{-1}(\beta - 1/2), & 1/2 < \beta \leq 1 - 2^{-\gamma/(\gamma-1)}, \\
(1 - (1 - \beta)^{1/\gamma})\gamma, & 1 - 2^{-\gamma/(\gamma-1)} \leq \beta < 1.
\end{cases}
$$
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(1 - (1 - \beta)^{1/\gamma})\gamma, & 1 - 2^{-\gamma/(\gamma-1)} \leq \beta < 1.
\end{cases}$$

• Detection boundary for $0 < \gamma \leq 1$:

$$\rho^*_\gamma(\beta) = 2(\beta - 1/2), \quad 1/2 < \beta < 1.$$ 

Note: The detection boundary is the same for all for $0 < \gamma \leq 1$!
Figure 3. Detection boundaries for GN testing problem, $\gamma \in \{1, 1.5, 2, 3\}$. 
• **Theorem:** (Donoho - Jin, 2004). For the higher criticism test statistic applied to the p-values $p_i \equiv P(GN_\gamma(0) > Y_i)$, $i = 1, \ldots, n$. Then the detection boundary $\rho_{HC,\gamma}$ for this procedure is the same as the efficient detection boundary:

$$\rho_{HC,\gamma}(\beta) = \rho^*_\gamma(\beta), \quad 1/2 < \beta < 1.$$
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$$\rho_{HC,\gamma}(\beta) = \rho^*(\beta), \quad 1/2 < \beta < 1.$$ 

• Similar theorem for $\chi^2_v$ mixtures.
5. Donoho - Jin Power heuristics

What part of the sample contributes to the power?

- When $\beta \in [3/4, 1)$, the strongest evidence against $H_0$ is found near the maximum of the observations; i.e. at the smallest $p$-values.
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- When $\beta \in (1/2, 3/4]$ other $p$–values beyond the smallest contribute to the power.
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- When $\beta \in (\frac{1}{2}, \frac{3}{4}]$ other $p$-values beyond the smallest contribute to the power.
- Since the higher criticism statistic $HC_n^*$ gives more weight to the smaller $p$-values, we expect it to have higher power for alternatives with $\beta \in [\frac{3}{4}, 1)$.
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- When $\beta \in (1/2, 3/4]$ other $p$-values beyond the smallest contribute to the power.

- Since the higher criticism statistic $HC_n^*$ gives more weight to the smaller $p$-values, we expect it to have higher power for alternatives with $\beta \in [3/4, 1)$.

- Since the Berk-Jones (supremum of pointwise likelihood ratios) statistic $BJ_n^+$ gives less weight to the very smallest $p$-values, we expect that it might have higher power for $\beta \in (1/2, 3/4]$. 

6. Walther’s weighted likelihood ratio statistic

Let

\[ \log LR_n(t) = \begin{cases} 
  n \{ F_n(t) \log \frac{F_n(t)}{t} + (1 - F_n(t)) \log \frac{1 - F_n(t)}{1 - t} \}, & \text{if } 0 < t < F_n(t) \\
  0, & \text{otherwise}
\end{cases} \]

Thus

\[ BJ_n^+ = \max_{1 \leq i \leq n/2} \log LR_{n,i} \]

where

\[ \log LR_{n,i} \equiv \log LR_n(p(i)) \]

\[ = \left\{ i \log \left( \frac{i}{np(i)} \right) + (n - i) \log \left( \frac{1 - i/n}{1 - p(i)} \right) \right\} 1\{p(i) < i/n\}. \]
Start with a uniform prior on $\beta \in [1/2, 1)$. Since the smallest $p$-value has most of the information for $\beta \in [3/4, 1)$, collapse the weight for this interval to weight $1/2$ on the interval $(0, p_{(1)}]$. For $\beta \in (1/2, 3/4)$, the most promising interval to detect alternatives with $r$ close to the detection boundary $\rho^*(\beta) = \beta - 1/2$ is the interval $(0, n^{-4r}]$. Thus given such a $\beta$ we will use the LR test on the interval $(0, t]$ with $t = n^{-4(\beta-1/2)}$. If $\beta \sim U(1/2, 3/4)$, then $t = n^{-4(\beta-1/2)}$ has density proportional to $1/t$ on $(1/n, 1]$. 
Approximation of the resulting posterior integral with the corresponding weighted sum of the LR at the \( p(i) \)'s, normalized by

\[
\sum_{i=2}^{n/2} i^{-1} \approx \log(n/3)
\]

yields the **Average Likelihood Ratio Statistic**

\[
ALR_n = \frac{1}{2} LR_{n,1} + \frac{1}{2} \sum_{i=2}^{n/2} \frac{1}{i \log(n/3)} LR_{n,i}
\]

where

\[
LR_{n,i} = \begin{cases} 
\left( \frac{i}{np(i)} \right)^i \left( \frac{1-i/n}{1-p(i)} \right)^{n-i}, & \text{if } p(i) < i/n, \\
1, & \text{if otherwise.}
\end{cases}
\]
Proposition. (Walther) $ALR_n$ attains the optimal detection boundary for the sparse normal means problem.
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- More systematic study of power properties of all these tests.
Vielen Dank!