Semiparametric Gaussian Copula Models: Progress and Problems

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Outline

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• 1: Bivariate Gaussian copula models
• 2: $d$–variate Gaussian Copula models
• 3: Recent progress and results
• 4: Questions and open problems
0. Basics: notation and facts

Notation:

- $\Theta \subset \mathbb{R}^q$, $q \geq 1$; $\mathcal{F} = \{\text{all distribution functions on } \mathbb{R}\}$.

- Copulas: $\{C_\theta : \theta \in \Theta\} = \{\text{parametric family of distribution functions on } [0, 1]^d \text{ with uniform marginal distributions } C_\theta(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j \text{ for } u_j \in (0, 1) \text{ and } j = 1, \ldots, d\}$.

- Semiparametric copula distribution functions and measures: 
  
  
  \begin{align*}
  F_{\theta,F_1,\ldots,F_d}(x_1, \ldots, x_d) &= C_\theta(F_1(x_1), \ldots, F_d(x_d)) \text{ for distribution functions } F_j \text{ on } \mathbb{R}, \\
  P_{\theta,F_1,\ldots,F_d}(A) &= \int_A dF_{\theta,F_1,\ldots,F_d}(x), \ A \in \mathcal{B}^d.
  \end{align*}

- Semiparametric copula model:
  
  \begin{align*}
  \mathcal{P} &= \{P_{\theta,F_1,\ldots,F_d} : \theta \in \Theta, \ F_j \in \mathcal{F}, \ j = 1, \ldots, d\}.
  \end{align*}
Main focus here: multivariate Gaussian copulas

\[ \Phi_\theta(x) = P_\theta(X \leq x) = \text{d.f. of } N_d(0, \Sigma(\theta)), \]

where

\[ \Sigma(\theta) = \begin{pmatrix}
1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1,d} \\
\rho_{12} & 1 & \rho_{23} & \cdots & \rho_{2,d} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\rho_{1,d} & \rho_{d-1,d} & \cdots & 1
\end{pmatrix} \]

and \( \rho_{i,j} \equiv \rho_{i,j}(\theta) \). Then

\[ C_\theta(u) = \Phi_\theta(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)), \]
\[ c_\theta(u) = \frac{\phi_\theta(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d))}{\prod_{j=1}^{d} \phi(\Phi^{-1}(u_j))}, \]

for \( u = (u_1, \ldots, u_d) \in (0,1)^d \), and ...
\[ F_{\theta,F_1,\ldots,F_d}(x_1,\ldots,x_d) = C_{\theta}(F_1(x_1),\ldots,F_d(x_d)), \quad \theta \in \Theta, \quad F_j \in \mathcal{F}, \]

and \( \mathcal{P}_d \) is a semiparametric Gaussian copula model based on \( c_\theta \).

Now suppose that we observe \( X_1,\ldots,X_n \) i.i.d. with probability distribution \( P_{\theta_0,F_{0,1},\ldots,F_{0,d}} \in \mathcal{P}_d \).

**Questions:**

- How well can we estimate \( \theta \in \Theta \)? (Lower bounds)
- Can we construct (rank-based) estimators achieving the lower bounds?
Since the model is invariant under monotone transformations on each axis, it is clear that the (multivariate) ranks are a maximal invariant.

More notation: let $X$ denote the $n \times d$ matrix with rows $X_1, \ldots, X_n$. Let $R(X) : \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^{n \times d}$ be the corresponding $n \times d$ matrix of ranks where $R = (R_{i,j})$ and

$$R_{i,j} = \text{the rank of } X_{i,j} \text{ among } \{X_{1,j}, \ldots, X_{n,j}\}, \quad j = 1, \ldots, d.$$

Hoff (2007) has shown that the ranks $R$ are partially sufficient in several senses, and it seems natural to try base inference procedures on them if possible.
1. Bivariate Gaussian copulas

Here \( d = 2 \) and \( \theta \in \Theta = (-1, 1) \). Klaassen and W (1997) showed:

- \( I_\theta(P_2) = (1 - \theta^2)^{-2} \).
- Normal margins are least favorable.
- \( \hat{\theta}_n = \) normal scores rank correlation coefficient is asymptotically efficient:
  \[
  \sqrt{n}(\hat{\theta}_n - \theta) \rightarrow_d N(0, (1 - \theta^2)^2).
  \]
- \( \hat{\theta}_n \) is asymptotically equivalent to the maximum pseudo likelihood estimator \( \hat{\theta}_{n}^{ple} \): \( \sqrt{n}(\hat{\theta}_n - \hat{\theta}_n^{ple}) = o_p(1) \) where
  \[
  \hat{\theta}_n^{ple} = \arg\max_{\theta \in \Theta} \ell_n(\theta, G_n, H_n)
  \]
  where \( G_n, H_n \), are the marginal empirical distribution functions of the data. (Note that \( \hat{\theta}_n^{ple} \) is also a function of the ranks.)
Here with $X_i = (Y_i, Z_i), \ i = 1, \ldots, n,$

$$\widehat{\theta}_n = \frac{n^{-1} \sum_{i=1}^{n} \Phi^{-1}(G_n^*(Y_i)) \Phi^{-1}(H_n^*(Z_i))}{n^{-1} \sum_{i=1}^{n} \Phi^{-1}\left(\frac{i}{n+1}\right)^2}$$

$$= \frac{n^{-1} \sum_{i=1}^{n} \Phi^{-1}\left(\frac{R_{i,1}}{n+1}\right) \Phi^{-1}\left(\frac{R_{i,2}}{n+1}\right)}{n^{-1} \sum_{i=1}^{n} \Phi^{-1}\left(\frac{i}{n+1}\right)^2}$$

**Asymptotic linearity:**

$$\sqrt{n} (\widehat{\theta}_n - \theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{\ell}_\theta(X_i) + o_p(1)$$

where

$$\tilde{\ell}_\theta(y, z) = I_\theta^{-1} \ell_\theta^*(y, z)$$

$$= \Phi^{-1}(G(y)) \Phi^{-1}(H(z)) - \frac{\theta}{2} \left(\Phi^{-1}(G(y))^2 + \Phi^{-1}(H(z))^2\right).$$
2. Multivariate Gaussian copulas, $d > 2$

- When $\Sigma(\theta)$ is unstructured (i.e. $\theta = (\rho_{1,2}, \rho_{1,3}, \ldots, \rho_{1,d}, \ldots, \rho_{d-1,d}) \in [-1, 1]^{d(d-1)/2}$), then the pseudo-likelihood estimator continues to be semiparametric efficient, as noted by Klaassen & W (1997), and Segers, von den Akker, Werker (2014).

- What if $d > 2$ and $\Sigma(\theta)$ is structured?

Examples:

- Example 1. (Exchangeable) $\Sigma(\theta) = (1 - \theta)I_d + \theta 11^T$ with $\theta \in [-1/(d + 1), 1)$. For example for $d = 4$

$$
\Sigma(\theta) = \begin{pmatrix}
1 & \theta & \theta & \theta \\
\theta & 1 & \theta & \theta \\
\theta & \theta & 1 & \theta \\
\theta & \theta & \theta & 1
\end{pmatrix}.
$$
• Example 2. (Circular) For $d = 4$, 

$$
\Sigma(\theta) = \begin{pmatrix}
1 & \theta & \theta^2 & \theta \\
\theta & 1 & \theta & \theta^2 \\
\theta^2 & \theta & 1 & \theta \\
\theta & \theta^2 & \theta & 1
\end{pmatrix}.
$$

• Example 3. (Toeplitz). Here $\Sigma = (\sigma_{i,j})$ with $\sigma_{i,i} = 1$ for all $i$, $\sigma_{i,j} = \theta_{|i-j|}$ for $\theta = (\theta_1, \theta_2, \ldots, \theta_{d-1}) \in (-1, 1)^{d-1}$. For example, with $d = 4$, 

$$
\Sigma(\theta) = \begin{pmatrix}
1 & \theta_1 & \theta_2 & \theta_3 \\
\theta_1 & 1 & \theta_1 & \theta_2 \\
\theta_2 & \theta_1 & 1 & \theta_1 \\
\theta_3 & \theta_2 & \theta_1 & 1
\end{pmatrix}.
$$
More background:

- Genest and Werker (2000): studied efficiency properties of pseudo-likelihood estimators for general semiparametric copula models:
  Conclusion: $\hat{\theta}_n^{ple}$ is not efficient in general for (non-Gaussian) copulas.

- Chen, Fan, and Tsyrennikov (2006) constructed semiparametric efficient estimators for general multivariate copula models using parametric sieve methods. Their estimators of $\theta$ are not based solely on the multivariate ranks.
Questions:

- Do Maximum Likelihood Estimators based on rank likelihoods achieve semiparametric efficiency for general multivariate copula models?

- Do alternative estimators based on ranks achieve semiparametric efficiency?

- Are the pseudo maximum likelihood estimators semiparametric efficient for structured Gaussian copula models?
For $\theta \in \Theta \subset \mathbb{R}^q$ with $q < d(d - 1)/2$, let

$$L(\theta; \mathbf{R})$$

denote the likelihood of the ranks $\mathbf{R}$,

$$L(\theta, \psi; \mathbf{X})$$

denote the likelihood of the data $\mathbf{X}$,

where $\psi \in \Psi$ denotes parameters for the marginal transformations. For fixed $\theta \in \Theta$, $\psi \in \Psi$ let

$$\lambda_{\mathbf{R}}(t) \equiv \log \frac{L(\theta + t/\sqrt{n}; \mathbf{R})}{L(\theta; \mathbf{R})},$$

$$\lambda_{\mathbf{X}}(t, s) \equiv \log \frac{L(\theta + t/\sqrt{n}, \psi + s/\sqrt{n}; \mathbf{X})}{L(\theta, \psi; \mathbf{X})}. $$
Theorem 1. (Hoff-Niu-W, 2014) Let \( \{F_{\theta,\psi}(x) : \theta \in \Theta, \psi \in \Psi\} \) be an absolutely continuous copula model where, for given \( \theta \) and \( t \) there exist \( \psi \) and \( s \) such that under i.i.d. sampling from \( F_{\theta,\psi} \). Suppose that:

1. \( \lambda_X(t, s) \) satisfies Local Asymptotic Normality (LAN):
   \[
   \lambda_X(t, s) \rightarrow_d Z
   \]

2. There exists an \( \mathbb{R} \)-measurable approximation \( \hat{\lambda}_X(t, s) \) such that
   \[
   \lambda_X(t, s) - \hat{\lambda}_X \rightarrow_p 0.
   \]

Then \( \lambda_R(t) \rightarrow_d Z \) under i.i.d. sampling from any population with copula \( C_\theta(\cdot) \) equal to that of \( F(\cdot; \theta, \psi) \) and arbitrary absolutely continuous marginal distributions.

Conclusion: To show that the local likelihood ratio of the ranks satisfies LAN (from which an information bound follows for procedures based on the ranks follows), we need to construct suitable rank-measurable approximations of the local likelihood ratios of the data for parametric submodels.
Let \( X_1, \ldots, X_n \) be i.i.d. from a member \( P_{\theta, \psi} \) of a collection of \( N_d(0, \Sigma_{\theta, \psi}) \) where \( \theta \) parameterizes the correlations and \( \psi \) are the variance parameters. Then

\[
\lambda_X(t, s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i^T A X_i + c(\theta, \psi, t, s) + o_p(1)
\]

where \( A = A_{t, s, \theta, \psi} \). A natural rank-based approximation is

\[
\hat{\lambda}_X(t, s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \hat{X}_i^T A \hat{X}_i + c(\theta, \psi, t, s)
\]

where

\[
\hat{X}_{i,j} \equiv \sqrt{Var(X_{i,j})} \Phi^{-1} \left( \frac{R_{i,j}}{n + 1} \right).
\]

This leads to the following theorem:
Theorem 2. (Hoff, Niu, & W, 2014) Let $X_1, \ldots, X_n$ be i.i.d. $N_d(0, C)$ where $C$ is a correlation matrix and let $\hat{X}_{i,j} = \Phi^{-1}(R_{i,j}/(n + 1))$. Let $A$ be a matrix such that the diagonal entries of $AC + A^T C$ are zero. Then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{\hat{X}_i^T A \hat{X}_i - X_i^T A X_i\} = o_p(1).$$

- The proof of Theorem 2 is based on some classical results of de Wet and Venter (1972).
- It remains to apply the results of Theorems 1 and 2 to the setting of Gaussian copulas:
Theorem 3. (Hoff, Niu, & W, 2014). Suppose that \( \{ \Sigma(\theta) : \theta \in \Theta \subset \mathbb{R}^q \} \) is a collection of positive definite correlation matrices such that \( \Sigma_{i,j}(\theta) \) is continuously differentiable with respect to each \( \theta_k, 1 \leq k \leq q \). If \( X_1, \ldots, X_n \) are i.i.d. \( P_{\theta,\psi} \) with absolutely continuous marginals and Gaussian copula \( C_\theta \) for some \( \theta \in \Theta \), then the local likelihood ratio of the ranks \( \lambda_R(t) \) satisfies LAN:

\[
\lambda_R(t) \to_d N(-(1/2)t^T I_{\theta\theta,\psi} t, t^T I_{\theta\theta,\psi} t)
\]

where \( I_{\theta\theta,\psi} \) is the information for \( \theta \) in the Gaussian model with correlation matrix \( \Sigma(\theta) \) and precisions \( \psi \).

Summary: Let \( B(\theta) \equiv \Sigma^{-1}(\theta) \). Then, for \( q = 1 \),

- The efficient score function \( \ell^*_\theta \) is, with \( \underline{y} = (\Phi^{-1}(F_1(x_1)), \ldots, \Phi^{-1}(F_d(x_d))) \):

\[
\ell^*_\theta(\underline{y}) = \ell_\theta - I_{\theta\psi} I_{\psi\psi}^{-1} \ell_\psi = \frac{1}{2} y^T \left\{ \psi \frac{\text{tr}(B_\theta C) B - \psi B_\theta}{d} \right\} y.
\]
The efficient influence function $\tilde{\ell}_\theta$ for $\theta$ is, with

$$y = (\Phi^{-1}(F_1(x_1), \ldots, \Phi^{-1}(F_d(x_d))):$$

$$\tilde{\ell}_\theta(y) = I_{\theta \theta}^{-1} \psi \ell^*_\theta(y),$$

where

$$I_{\theta \theta} \psi = (1/2) \{ \text{tr}(B_\theta CB_\theta C) - \text{tr}(B_\theta C)^2/d \}.$$ 

Consequences:

- No information concerning $\theta$ is lost (asymptotically) by reducing to the ranks $R$.

- Gaussian marginals are least favorable.

- The information bounds for estimation of $\theta$ in such a Gaussian copula model are given in terms of $I_{\theta \theta}^{-1} \psi$. 
The efficient influence function $\tilde{\ell}_\theta(x)$ can be shown to be

$$\tilde{\ell}_\theta(x) = I_{\theta\theta}^{-1} \left\{ \dot{\ell}_\theta(x) - I_{\theta\psi} \tilde{\ell}_\psi(x) \right\}$$

The influence function of the pseudo likelihood estimator is given by

$$\psi_\theta(x) = I_{\theta\theta}^{-1} \left( \dot{\ell}_\theta(x) - \sum_{j=1}^{d} W_j(x_j) \right)$$

where

$$W_j(x_j) = \int_{(0,1)^d} \left( \frac{\partial^2}{\partial \theta \partial u_j} \log c_\theta(u) \right) \left( 1\{ \Phi(x_j) \leq u_j \} - u_j \right) c_\theta(u) du.$$ 

**Corollary:** The maximum pseudo likelihood estimator is semi-parametric efficient if

$$\sum_{j=1}^{d} W_j(x_j) = \frac{1}{2} \text{tr} \left( B \Sigma_\theta \{ I - \text{diag}(x \circ x) \} \right).$$
When $q = 1$ (and then $\psi \in \mathbb{R}$), this simplifies to

$$
\tilde{\ell}_\psi(x) = \frac{1}{d} \sum_{j=1}^{d} (1 - x_j^2).
$$
Examples, continued:

- Example 1. (Exchangeable) $\Sigma(\theta) = (1 - \theta)I_d + \theta 11^T$. For $d = 4$, calculation yields

\[
I_{\theta \theta \cdot \psi}^{-1} = \frac{1}{6} (1 + 2\theta - 3\theta^2),
\]

\[
\bar{\ell}_\theta(x) = \frac{1}{12} \left\{ 2 \sum_{1 \leq i < j \leq 4} x_i x_j - 3\theta \sum_{j=1}^{4} x_j^2 \right\}, \quad \text{and}
\]

\[
-I_{\theta \psi} \bar{\ell}_\psi(x) = \frac{6\theta}{1 + 2\theta - 3\theta^2} \frac{1}{4} \sum_{j=1}^{4} (x_j^2 - 1)
\]

\[
= \frac{3\theta/2}{1 + 2\theta - 3\theta^2} \sum_{j=1}^{4} (x_j^2 - 1) = \sum_{j=1}^{4} W_j(x_j),
\]

so the pseudo-likelihood estimator is semiparametric efficient.
Figure 1, Example 1: Information bounds and Monte-carlo variance of p-mle: red, \( n = 800 \).
Example 2. (Circular) For $d = 4$, calculation yields

\[ I_{\theta \theta, \psi} = \frac{4}{(1 - \theta^2)^2}, \]

\[ \tilde{\ell}_\theta(x) = \frac{1}{8(1 - \theta^2)} \left\{ (1 + \theta^2) \sum_{j=i+1,i+3} x_i x_j \right. \]

\[ \left. - 2\theta \sum_{j=1}^{4} x_j^2 - 2\theta \sum_{j=i+2} x_i x_j \right\}, \text{ and} \]

\[ -I_{\theta \psi} \tilde{\ell}_\psi(x) = \text{a complicated quadratic in } x_j \text{'s and cubic in } \theta \]

\[ \neq \sum_{j=1}^{4} W_j(x_j) = -\frac{\theta}{1 - \theta^2} \sum_{j=1}^{4} (x_j^2 - 1). \]

so the pseudo-likelihood estimator is not semiparametric efficient.
Figure 1, Example 2: Information bound and variance of p-mle
Figure 2, Example 2: Difference, variance of p-mle and Information bound
Summary:

- Information bounds for (structured) multivariate Gaussian models are available and computable.
- Gaussian marginal distributions are least favorable.
- The pseudo likelihood estimator is not always semiparametric efficient (but perhaps not missing efficiency by much).

Questions:

- Can we construct rank-based semiparametric efficient estimators?
- Are the pseudo likelihood estimators sometimes seriously inefficient?

Segers, van den Akker, and Werker (2014) give affirmative answers to both questions!
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**Rank-based semiparametric efficient estimators:**

via a “one-step” method:

- Start with a $\sqrt{n}$–consistent rank based estimator $\hat{\theta}_n^0$; e.g. the pseudo likelihood estimator $\hat{\theta}_n^{ple}$.

- Construct the natural one-step estimator starting from $\hat{\theta}_n^0$ and based on the efficient score function $\ell_\theta^*$.
Recent progress and results

Inefficiency of pseudo likelihood estimator $\widehat{\theta}_{n}^{ple}$:

Example 3: (Toeplitz correlation model) Suppose that $\theta = (\theta_{1}, \ldots, \theta_{d-1}) \in (-1,1)^{d-1}$ and $\Sigma = (\sigma_{i,j})_{i,j=1}^{d} = (\sigma_{i,j}(\theta))$ where $\sigma_{i,i} = 1$ and $\sigma_{i,j}(\theta) = \theta_{|i-j|}$ for $j \neq i$. For example: when $d = 3$, $\theta = (\theta_{1}, \theta_{2}) \in (-1,1)^{2}$ and

$$\Sigma(\theta) = \begin{pmatrix} 1 & \theta_{1} & \theta_{2} \\ \theta_{1} & 1 & \theta_{1} \\ \theta_{2} & \theta_{1} & 1 \end{pmatrix};$$

when $d = 4$, $\theta = (\theta_{1}, \theta_{2}, \theta_{3}) \in (-1,1)^{3}$ and

$$\Sigma(\theta) = \begin{pmatrix} 1 & \theta_{1} & \theta_{2} & \theta_{3} \\ \theta_{1} & 1 & \theta_{1} & \theta_{2} \\ \theta_{2} & \theta_{1} & 1 & \theta_{1} \\ \theta_{3} & \theta_{2} & \theta_{1} & 1 \end{pmatrix}.$$
Recent progress and results

- For $d = 3$ the Pseudo-Likelihood Estimator (PLE) $\hat{\theta}_{n}^{ple}$ is semiparametric efficient.

- For $d = 4$, $\hat{\theta}_{n}^{ple}$ is not efficient, and some times severely so. When $\theta = (0.494546, -0.450276, -0.0846249)$, the asymptotic relative efficiencies of the PLE with respect to the information bound are 

  \[(18.3\%, 19.8\%, 96.9\%).\]

- The PLE is semiparametric efficient for a large class of “factor models”: if $\theta$ is a $d \times q$ matrix, $q < d$, $\Theta$ is an open subset of $\{\theta \in \mathbb{R}^{d\times q} : (\theta\theta^T)_{jj} < 1, j = 1, \ldots, d\}$ and 

  \[\Sigma(\theta) \equiv \theta\theta^T + (I_d - \text{diag}(\theta\theta^T)).\]
4: Questions and open problems

- Semiparametric efficient estimation of the marginal distributions?
  - Can we improve on the marginal empirical distribution functions? (Apparently not known even for bivariate Gaussian copula model?)
  - Asymptotic behavior of the sieve estimators of Chen, Fan, and Tsyrennikov (2006)?

- Asymptotic behavior of the MLE’s of $\theta$ based on the rank likelihood. (Rank likelihood is difficult to compute!)

- Rank-based semiparametric efficient estimators of $\theta$ for non-Gaussian copula’s?

- Asymptotic theory for P. Hoff’s “extended rank likelihood” (Hoff 2007, 2008)?

- What happens under model miss-specification? (Remember David X. Li (2000)!)
Selected references:

- Hoff, Niu, and W (2014). Bernoulli
- Klaassen and W (1997). Bernoulli
Cautions:

Xièxiè!