Empirical Process Theory for Statistics

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- Lecture Outline:
  1. Introduction, history, selected examples.
  2. Some basic inequalities and Glivenko-Cantelli theorems.
  3. Using the Glivenko-Cantelli theorems: first applications.
  4. Donsker theorems and some inequalities.
  5. Peeling methods and rates of convergence.
  6. Some useful preservation theorems.
Based on Courses given at Torgnon, Cortona, and Delft (2003-2005). Notes available at:

Part I: Introduction, history, selected examples

• 1. Classical empirical processes
• 2. Modern empirical processes
• 3. Some examples
1. Classical empirical processes. Suppose that:

- $X_1, \ldots, X_n$ are i.i.d. with d.f. $F$ on $\mathbb{R}$.
- $F_n(x) = n^{-1} \sum_{i=1}^{n} 1[X_i \leq x]$, the empirical distribution function.
- $\{Z_n(x) \equiv \sqrt{n}(F_n(x) - F(x)) : x \in \mathbb{R}\}$, the empirical process.

Two classical theorems:

**Theorem 1.** (Glivenko-Cantelli, 1933).

$$ ||F_n - F||_{\infty} \equiv \sup_{-\infty < x < \infty} |F_n(x) - F(x)| \to_{a.s.} 0. $$

**Theorem 2.** (Donsker, 1952).

$$ Z_n \Rightarrow Z \equiv \mathcal{U}(F) \text{ in } D(\mathbb{R}, ||\cdot||_{\infty}). $$
where $U$ is a standard Brownian bridge process on $[0,1]$; i.e. $U$ is a zero-mean Gaussian process with covariance

$$E(U(s)U(t)) = s \wedge t - st, \quad s, t \in [0,1].$$

This means that we have

$$Eg(Z_n) \to Eg(Z)$$

for any bounded, continuous function $g : D(\mathbb{R}, \| \cdot \|_{\infty}) \to \mathbb{R}$ and

$$g(Z_n) \to_d g(Z)$$

for any continuous function $g : D(\mathbb{R}, \| \cdot \|_{\infty}) \to \mathbb{R}$ (ignoring measurability issues).
2. General empirical processes (indexed by functions)
Suppose that:

• \(X_1, \ldots, X_n\) are i.i.d. with probability measure \(P\) on \((\mathcal{X}, \mathcal{A})\).

• \(P_n = n^{-1} \sum_{i=1}^{n} \delta_{X_i}\), the empirical measure; here

\[
\delta_x(A) = 1_A(x) = \begin{cases} 
1, & x \in A, \\
0, & x \in A^c
\end{cases} \quad \text{for} \quad A \in \mathcal{A}.
\]

Hence we have

\[
P_n(A) = n^{-1} \sum_{i=1}^{n} 1_A(X_i), \quad \text{and} \quad P_n(f) = n^{-1} \sum_{i=1}^{n} f(X_i).
\]

• \(\mathcal{G}_n(f) \equiv \sqrt{n}(P_n(f) - P(f)) : f \in \mathcal{F} \subset L_2(P)\}, \) the empirical process indexed by \(\mathcal{F}\)
Note that the classical case corresponds to:

- \((\mathcal{X}, \mathcal{A}) = (\mathbb{R}, \mathcal{B}).\)
- \(\mathcal{F} = \{1(\cdot) : t \in \mathbb{R}\}.\)

Then

\[
P_n(1_{(-\infty, t]}) = n^{-1} \sum_{i=1}^{n} 1_{(-\infty, t]}(X_i) = \mathbb{F}_n(t),
\]

\[
P(1_{(-\infty, t]}) = F(t),
\]

\[
\mathbb{G}_n(1_{(-\infty, t]}) = \sqrt{n}(p_n - P)(1_{(-\infty, t]} = \sqrt{n}(\mathbb{F}_n(t) - F(t))
\]

\[
\mathbb{G}(1_{(-\infty, t]}) = \mathbb{U}(F(t)).
\]
Two central questions for the general theory:

A. For what classes of functions $\mathcal{F}$ does a natural generalization of the Glivenko-Cantelli theorem hold? That is, for what classes $\mathcal{F}$ do we have

$$\|\mathbb{P}_n - P\|_\mathcal{F}^* \to_{a.s.} 0$$

If this convergence holds, then we say that $\mathcal{F}$ is a $P$–Glivenko-Cantelli class of functions.

B. For what classes of functions $\mathcal{F}$ does a natural generalization of Donsker’s theorem hold? That is, for what classes $\mathcal{F}$ do we have

$$\mathbb{G}_n \Rightarrow \mathbb{G}_P \text{ in } \ell^\infty(\mathcal{F})?$$

If this convergence holds, then we say that $\mathcal{F}$ is a $P$–Donsker class of functions.
Here $\mathbb{G}_P$ is a $0$–mean $P$–Brownian bridge process with uniformly-continuous sample paths with respect to the semi-metric $\rho_P(f, g)$ defined by
\[
\rho^2_P(f, g) = \text{Var}_P(f(X) - g(X)),
\]
$\ell^\infty(\mathcal{F})$ is the space of all bounded, real-valued functions $z$ from $\mathcal{F}$ to $\mathbb{R}$:
\[
\ell^\infty(\mathcal{F}) = \left\{ z : \mathcal{F} \mapsto \mathbb{R} \left| \|z\|_{\mathcal{F}} = \sup_{f \in \mathcal{F}} |z(f)| < \infty \right. \right\},
\]
and
\[
E\{\mathbb{G}_P(f)\mathbb{G}_P(g)\} = P(fg) - P(f)P(g).
\]
3. Some Examples

A commonly occurring problem in statistics: we want to prove consistency or asymptotic normality of some statistic which is not a sum of independent random variables, but which can be related to a natural sum of random functions indexed by a parameter in a suitable (metric) space.

**Example 1.** Suppose that $X_1, \ldots, X_n$ are i.i.d. real-valued with $E|X_1| < \infty$, and let $\mu = E(X_1)$. Consider the absolute deviations about the sample mean,

$$D_n = \mathbb{P}_n|X - \bar{X}_n| = n^{-1} \sum_{i=1}^{n} |X_i - \bar{X}_n|.$$

Since $\bar{X}_n \to a.s. \mu$, we know that for any $\delta > 0$ we have $\bar{X} \in [\mu - \delta, \mu + \delta]$ for all sufficiently large $n$ almost surely. Thus we see that if we define

$$D_n(t) \equiv \mathbb{P}_n|x - t| = n^{-1} \sum_{i=1}^{n} |X_i - t|,$$
then $D_n = D_n(\bar{X}_n)$ and study of $D_n(t)$ for $t \in [\mu - \delta, \mu + \delta]$ is equivalent to study of the empirical measure $\mathbb{P}_n$ indexed by the class of functions

$$\mathcal{F}_\delta = \{ x \mapsto |x - t| \equiv f_t(x) : t \in [\mu - \delta, \mu + \delta] \}.$$

To show that $D_n \rightarrow_{a.s.} d \equiv E|X - \mu|$, we write

$$D_n - d = \mathbb{P}_n|X - \bar{X}_n| - P|X - \mu| \leq \sup_{t: |t - \mu| \leq \delta} |(\mathbb{P}_n - P)|X - t|| = \sup_{f \in \mathcal{F}_\delta} |(\mathbb{P}_n - P)(f)| \rightarrow_{a.s.} 0 \quad (3)$$

if $\mathcal{F}_\delta$ is $P$–Glivenko-Cantelli.
But convergence of the second term in (2) is easy: by the triangle inequality

\[ II_n = |P|X - \overline{X}_n| - P|X - \mu|| \leq P|\overline{X}_n - \mu| = |\overline{X}_n - \mu| \rightarrow_{a.s.} 0. \]

How to prove (3)? Consider the functions \( f_1, \ldots, f_m \in \mathcal{F}_\delta \) given by

\[ f_j(x) = |x - (\mu - \delta(1 - j/m)|, \quad j = 0, \ldots, 2m. \]

For this finite set of functions we have

\[ \max_{0 \leq j \leq 2m} |(\mathbb{P}_n - P)(f_j)| \rightarrow_{a.s.} 0 \]

by the strong law of large numbers applied \( 2m + 1 \) times. Furthermore ...
it follows that for $t \in [\mu - \delta(1 - j/m), \mu - \delta(1 - (j + 1)/m)]$ the functions $f_t(x) = |x - t|$ satisfy (picture!)

$L_j(x) \equiv f_{j/m}(x) \land f_{(j+1)/m}(x) \leq f_t(x) \leq f_{j/m}(x) \lor f_{(j+1)/m}(x) \equiv U_j(x)$

where

$$U_j(x) - f_t(x) \leq \frac{1}{m}, \quad f_t(x) - L_j(x) \leq \frac{1}{m}, \quad U_j(x) - L_j(x) \leq \frac{1}{m}.$$  

Thus for each $m$

$$\|\mathbb{P}_n - P\|_{F_\delta} \equiv \sup_{f \in \mathcal{F}_\delta} |(\mathbb{P}_n - P)(f)| \leq \max \left\{ \max_{0 \leq j \leq 2m} |(\mathbb{P}_n - P)(U_j)|, \max_{0 \leq j \leq 2m} |(\mathbb{P}_n - P)(L_j)| \right\} + \frac{1}{m} \rightarrow a.s. \quad 0 + \frac{1}{m}$$

Taking $m$ large shows that (3) holds.
This is a bracketing argument, and generalizes easily to yield a quite general bracketing Glivenko-Cantelli theorem.

How to prove $\sqrt{n}(D_n - d) \to_d$? We write

$$\sqrt{n}(D_n - d) = \sqrt{n}(\mathbb{P}_n|X - \bar{X}_n| - P|X - \mu|)$$

$$= \sqrt{n}(\mathbb{P}_n|X - \mu| - P|X - \mu|)$$

$$+ \sqrt{n}(P|X - \bar{X}_n| - P|X - \mu|)$$

$$+ \sqrt{n}(\mathbb{P}_n - P)(|X - \bar{X}_n|) - \sqrt{n}(\mathbb{P}_n - P)(|X - \mu|)$$

$$= \mathcal{G}_n(|X - \mu|) + \sqrt{n}(H(\bar{X}_n) - H(\mu))$$

$$+ \mathcal{G}_n(|X - \bar{X}_n| - |X - \mu|)$$

$$= \mathcal{G}_n(|X - \mu|) + H'(\mu)(\bar{X}_n - \mu)$$

$$+ \sqrt{n}(H(\bar{X}_n) - H(\mu) - H'(\mu)(\bar{X}_n - \mu))$$

$$+ \mathcal{G}_n(|X - \bar{X}_n| - |X - \mu|)$$

$$\equiv \mathcal{G}_n(|X - \mu| + H'(\mu)(X - \mu)) + I_n + II_n$$

where ...
\[ H(t) \equiv P|X - t|, \]
\[ I_n \equiv \sqrt{n}(H(\overline{X}_n) - H(\mu) - H'(\mu)(\overline{X}_n - \mu)), \]
\[ II_n \equiv \mathbb{G}_n(|X - \overline{X}_n|) - \mathbb{G}_n(|X - \mu|) \]
\[ = \mathbb{G}_n(|X - \overline{X}_n| - |X - \mu|) \]
\[ = \mathbb{G}_n(f_{\overline{X}_n} - f_\mu). \]

Here \( I_n \to_p 0 \) if \( H(t) \equiv P|X - t| \) is differentiable at \( \mu \). The second term

\[ II_n \equiv \mathbb{G}_n(f_{\overline{X}_n} - f_\mu) \to_p 0 \]

if \( \mathcal{F}_\delta \) is a Donsker class of functions! This is a consequence of asymptotic equicontinuity of \( \mathbb{G}_n \) over the class \( \mathcal{F} \): for every \( \epsilon > 0 \)

\[ \lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup_{f, g: \rho_P(f, g) \leq \delta} Pr^*(\sup_{|\mathbb{G}_n(f) - \mathbb{G}_n(g)| > \epsilon}) = 0. \]
Example 2. Copula models: the pseudo-MLE.

Let $c_\theta(u_1, \ldots, u_p)$ be a copula density with $\theta \subset \Theta \subset \mathbb{R}^q$. Suppose that $X_1, \ldots, X_n$ are i.i.d. with density

$$f(x_1, \ldots, x_p) = c_\theta(F_1(x_1), \ldots, F_p(x_p)) \cdot f_1(x_1) \cdots f_p(x_p)$$

where $F_1, \ldots, F_p$ are absolutely continuous d.f.'s with densities $f_1, \ldots, f_p$.

Let

$$F_{n,j}(x_j) \equiv n^{-1} \sum_{i=1}^n 1\{X_{i,j} \leq x_j\}, \quad j = 1, \ldots, p$$

be the marginal empirical d.f.'s of the data. Then a natural pseudo-likelihood function is given by

$$l_n(\theta) \equiv \mathbb{P}n \log c_\theta(F_{n,1}(x_1), \ldots, F_{n,p}(x_p)).$$
Thus it seems reasonable to define the pseudo-likelihood estimator $\hat{\theta}_n$ of $\theta$ by the $q$–dimensional system of equations

$$\Psi_n(\hat{\theta}_n) = 0$$

where

$$\Psi_n(\theta) \equiv \mathbb{P}_n(\ell_\theta(\theta; F_{n,1}(x_1), \ldots, F_{n,p}(x_p)))$$

and where

$$\ell_\theta(\theta; u_1, \ldots, u_p) \equiv \nabla_\theta \log c_\theta(u_1, \ldots, u_p).$$

We also define $\Psi(\theta)$ by

$$\Psi(\theta) \equiv P_0(\ell_\theta(\theta, F_1(x_1), \ldots, F_p(x_p))).$$
Then we expect that

\[ 0 = \Psi_n(\hat{\theta}_n) = \Psi_n(\theta_0) - \left\{-\dot{\Psi}_n(\theta^*_n)\right\}(\hat{\theta}_n - \theta_0) \quad (4) \]

where

\[ \psi_n(\theta_0) = \mathbb{P}_n \ell_{\theta}(\theta_0, F_{n,1}(x_1), \ldots, F_{n,p}(x_p)), \]

and

\[ -\dot{\psi}_n(\theta^*_n) = -\mathbb{P}_n \ell_{\theta,\theta}(\theta^*_n, F_{n,1}(x_1), \ldots, F_{n,p}(x_p)) \]
\[ \rightarrow P_0(\ell_{\theta,\theta}(\theta_0, F_1(x_1), \ldots, F_p(x_p))) \]
\[ \equiv B \equiv I_{\theta\theta} \quad (5) \]

a \( q \times q \) matrix. On the other hand ...
\[ \sqrt{n} \Psi_n(\theta_0) = \sqrt{n} \mathbb{P}_n \dot{\ell}_\theta(\theta_0, F_{n,1}(x_1), \ldots, F_{n,p}(x_p)) \]

where

\[
\dot{\ell}_\theta(\theta_0, F_{n,1}(x_1), \ldots, F_{n,p}(x_p)) \\
= \dot{\ell}_\theta(\theta_0, F_1(x_1), \ldots, F_p(x_p)) \\
+ \sum_{j=1}^p \ddot{\ell}_{\theta,j}(\theta_0, u_1^*, \ldots, u_p^*) \cdot (F_{n,j}(x_j) - F_j(x_j)),
\]

and where \(|u_j^*(x_j) - F_j(x_j)| \leq |F_{n,j}(x_j) - F_j(x_j)|\) for \(j = 1, \ldots, p\).

Thus we expect that
\[ \sqrt{n} \Psi_n(\theta_0) \]
\[ = \sqrt{n}\mathbb{P}_n(\hat{\ell}_\theta(\theta_0, F_{n,1}(x_1), \ldots, F_{n,p}(x_p))) \]
\[ \equiv \mathcal{G}_n \left( \hat{\ell}_\theta(\theta_0, F_1(x_1), \ldots, F_p(x_p)) \right) \]
\[ + \mathbb{P}_n \left( \sum_{j=1}^{p} \hat{\ell}_{\theta,j}(\theta_0, u_1^*, \ldots, u_p^*) \cdot \sqrt{n}(F_{n,j}(x_j) - F_j(x_j)) \right) \]
\[ = \mathcal{G}_n \left( \hat{\ell}_\theta(\theta_0, F_1(x_1), \ldots, F_p(x_p)) \right) \]
\[ + P_0 \left( \sum_{j=1}^{p} \hat{\ell}_{\theta,j}(\theta_0, u_1^*, \ldots, u_p^*) \cdot \sqrt{n}(F_{n,j}(x_j) - F_j(x_j)) \right) \]
\[ + (\mathbb{P}_n - P_0) \left( \sum_{j=1}^{p} \hat{\ell}_{\theta,j}(\theta_0, u_1^*, \ldots, u_p^*) \cdot \sqrt{n}(F_{n,j}(x_j) - F_j(x_j)) \right). \]

In this last display the third term will be negligible (via asymptotic equicontinuity!) and the second term can be rewritten as
\[ P_0 \left( \sum_{j=1}^{p} \hat{\ell}_{\theta,j}(\theta_0, u_1^*, \ldots, u_p^*) \cdot \sqrt{n}(F_{n,j}(x_j) - F_j(x_j)) \right) \]

\[ = \sum_{j=1}^{p} P_0 \hat{\ell}_{\theta,j}(\theta_0, u_1^*(x_1), \ldots, u_p^*(x_p)) \cdot \sqrt{n}(F_{n,j}(x_j) - F_j(x_j)) \]

\[ = \mathcal{G}_n \left( \sum_{j=1}^{p} \int_{R^p} \hat{\ell}_{\theta,j}(\theta_0, F_1(x_1), \ldots, F_p(x_p)) \cdot \left( 1\{X_j \leq x_j\} - F_j(x_j) \right) dC_{\theta}(F_1(x_1), \ldots, F_p(x_p)) \right) \]

\[ = \mathcal{G}_n \left( \sum_{j=1}^{p} \int_{[0,1]^p} \hat{\ell}_{\theta,j}(\theta_0, u_1, \ldots, u_p) \cdot \left( 1\{F_j(X_j) \leq u_j\} - u_j \right) dC_{\theta}(u_1, \ldots, u_p) \right) \]

\[ = \mathcal{G}_n \left( \sum_{j=1}^{p} W_j(X_j) \right) \]
**Example 3.** Kendall’s function.
Suppose that \((X_1,Y_1), \ldots, (X_n,Y_n), \ldots\) are i.i.d. \(F_0\) on \(\mathbb{R}^2\), and let \(F_n\) denote their (classical) empirical distribution function
\[
F_n(x,y) = \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty,x] \times (-\infty,y]}(X_i,Y_i).
\]
Consider the empirical distribution function of the random variables \(F_n(X_i,Y_i), i = 1, \ldots, n:\)
\[
K_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1_{[F_n(X_i,Y_i) \leq t]}, \quad t \in [0,1].
\]
As in example 1, the random variables \(\{F_n(X_i,Y_i)\}_{i=1}^{n}\) are dependent, and we are already studying a stochastic process indexed by \(t \in [0,1]\). The empirical process method leads to study of the process \(K_n\) indexed by both \(t \in [0,1]\) and \(F \in \mathcal{F}_2\), the class of all distribution functions \(F\) on \(\mathbb{R}^2:\)
\[
K_n(t,F) \equiv \frac{1}{n} \sum_{i=1}^{n} 1_{[F(X_i,Y_i) \leq t]} = \mathbb{P}_n 1_{[F(X,Y) \leq t]}
\]
with $t \in [0, 1]$ and $F \in \mathcal{F}_2$ ... or the smaller set 

$$
\mathcal{F}_{2,\delta} = \{ F \in \mathcal{F}_2 : \| F - F_0 \|_\infty \leq \delta \}.
$$
Example 4. Completely monotone densities. Consider the class \( \mathcal{P} \) of completely monotone densities \( p_G \) given by

\[
p_G(x) = \int_0^\infty z \exp(-zx) dG(z)
\]

where \( G \) is an arbitrary distribution function on \( \mathbb{R}^+ \). Consider the maximum likelihood estimator \( \hat{p} \) of \( p \in \mathcal{P} \): i.e.

\[
\hat{p} \equiv \arg\max_{p \in \mathcal{P}} \mathbb{P}_n \log(p).
\]

Question: Is \( \hat{p} \) Hellinger consistent? That is, do we have

\[
h(\hat{p}_n, p_0) \to_{a.s.} 0?
\]
Part II: Some basic inequalities and Glivenko-Cantelli theorems

1. Tools for consistency: two basic inequalities.
2. Tools for consistency:
   a further basic inequality for convex $\mathcal{P}$.
3. More basic inequalities:
   least squares estimators; penalized ML.
4. Glivenko-Cantelli theorems.
1. Tools for consistency: two basic inequalities

**Density estimation** Suppose that:

- $\mathcal{P}$ is a class of densities with respect to a fixed $\sigma$–finite measure $\mu$ on a measurable space $(\mathcal{X}, \mathcal{A})$.

- Suppose that $X_1, \ldots, X_n$ are i.i.d. $P_0$ with density $p_0 \in \mathcal{P}$.

- Then the Maximum Likelihood Estimator (MLE) for the class $\mathcal{P}$ is

$$\hat{p}_n \equiv \arg\max_{p \in \mathcal{P}} P_n \log(p).$$
Here are two “basic inequalities” for density estimation.

**Proposition 1.1.** (Van de Geer). Suppose that $\hat{p}_n$ maximizes $P_n \log(p)$ over $\mathcal{P}$. then

$$h^2(\hat{p}_n, p_0) \leq (P_n - P_0) \left( \sqrt{\frac{\hat{p}_n}{p_0}} - 1 \right) 1\{p_0 > 0\}.$$ 

**Proposition 1.2.** (Birgé and Massart). If $\hat{p}_n$ maximizes $P_n \log(p)$ over $\mathcal{P}$, then

$$h^2(\frac{\hat{p}_n + p_0}{2}, p_0) \leq (P_n - P_0) \left( \frac{1}{2} \log \left( \frac{\hat{p}_n + p_0}{2p_0} \right) 1[p_0 > 0] \right),$$

and

$$h^2(\hat{p}_n, p_0) \leq 24 h^2 \left( \frac{\hat{p}_n + p_0}{2}, p_0 \right).$$
• Proposition 1.1 leads to the class of functions
\[ \mathcal{F} = \left\{ \left( \sqrt{\frac{p}{p_0}} - 1 \right) : \ p \in \mathcal{P} \right\} . \]
and the question: Is \( \mathcal{F} \) a \( P_0 \)-Glivenko class?

• Proposition 1.2 leads to the class of functions
\[ \mathcal{F} = \left\{ \left( \frac{1}{2} \log \left( \frac{p + p_0}{2p_0} \right) 1_{[p_0 > 0]} \right) : \ p \in \mathcal{P} \right\} . \]
and the question: Is \( \mathcal{F} \) a \( P_0 \)-Glivenko class?
Proof, proposition 1.1: Since \( \hat{p}_n \) maximizes \( P_n \log p \),

\[
0 \leq \frac{1}{2} \int_{[p_0 > 0]} \log \left( \frac{\hat{p}_n}{p_0} \right) dP_n
\]

\[
\leq \int_{[p_0 > 0]} \left( \sqrt{\frac{\hat{p}_n}{p_0}} - 1 \right) dP_n
\]

since \( \log(1 + x) \leq x \)

\[
= \int_{[p_0 > 0]} \left( \sqrt{\frac{\hat{p}_n}{p_0}} - 1 \right) d(P_n - P_0)
\]

\[
+ P_0 \left( \sqrt{\frac{\hat{p}_n}{p_0}} - 1 \right) 1\{p_0 > 0\}
\]

\[
= \int_{[p_0 > 0]} \left( \sqrt{\frac{\hat{p}_n}{p_0}} - 1 \right) d(P_n - P_0) - h^2(\hat{p}_n, p_0)
\]

where the last equality follows by direct calculation and the definition of the Hellinger metric \( h \). \( \square \)
Proof, Proposition 1.2: By concavity of log,
\[
\log \left( \frac{\hat{p}_n + p_0}{2p_0} \right) 1_{[p_0 > 0]} \geq \frac{1}{2} \log \left( \frac{\hat{p}_n}{p_0} \right) 1_{[p_0 > 0]}.
\]
Thus
\[
0 \leq \mathbb{P}_n \left( \frac{1}{4} \log \left( \frac{\hat{p}_n}{p_0} \right) 1_{[p_0 > 0]} \right) \leq \mathbb{P}_n \left( \frac{1}{2} \log \left( \frac{\hat{p}_n + p_0}{2p_0} \right) 1_{[p_0 > 0]} \right)
\]
\[
= \left( \mathbb{P}_n - P_0 \right) \left( \frac{1}{2} \log \left( \frac{\hat{p}_n + p_0}{2p_0} \right) 1_{[p_0 > 0]} \right)
\]
\[
+ P_0 \left( \frac{1}{2} \log \left( \frac{\hat{p}_n + p_0}{2p_0} \right) 1_{[p_0 > 0]} \right)
\]
\[
= \left( \mathbb{P}_n - P_0 \right) \left( \frac{1}{2} \log \left( \frac{\hat{p}_n + p_0}{2p_0} \right) 1_{[p_0 > 0]} \right) - \frac{1}{2} K(P_0, (\hat{P}_n + P_0)/2)
\]
\[
\leq \left( \mathbb{P}_n - P_0 \right) \left( \frac{1}{2} \log \left( \frac{\hat{p}_n + p_0}{2p_0} \right) 1_{[p_0 > 0]} \right) - h^2(P_0, (\hat{P}_n + P_0)/2).
\]
where we used Exercise 1.2 at the last step. The second claim
follows from Exercise 1.4.

Exercise 1.2: (Pinsker inequalities)
(a) $K(P, Q) \geq 2h^2(P, Q) = \int [\sqrt{p} - \sqrt{q}]^2 d\mu.
(b) K(P, Q) \geq (1/2) (\int |p - q| d\mu)^2 = 2d_{TV}^2(P, Q).

Exercise 1.4:

$$2h^2(P, (P + Q)/2) \leq h^2(P, Q) \leq 12h^2(P, (P + Q)/2).$$

Corollary 1.1. (Hellinger consistency of MLE). Suppose that either

$$\{ (\sqrt{p/p_0} - 1) \mathbb{1}_{p_0 > 0} : p \in \mathcal{P} \}, \text{ or } \left\{ \frac{1}{2} \log \left( \frac{p + p_0}{2p_0} \right) \mathbb{1}_{[p_0 > 0]} : p \in \mathcal{P} \right\}$$

is a $P_0$–Glivenko-Cantelli class. Then $h(\hat{p}_n, p_0) \to_{a.s.} 0.$
2. Tools for consistency: a further basic inequality.

- For $0 < \alpha \leq 1$, let $\varphi_\alpha(t) = (t^\alpha - 1)/(t^\alpha + 1)$ for $t \geq 0$, $\varphi(t) = -1$ for $t < 0$. Thus $\varphi_\alpha$ is bounded and continuous for each $\alpha \in (0, 1]$.

- For $0 < \beta < 1$ define
  \[
  h_\beta^2(p, q) \equiv 1 - \int p^\beta q^{1-\beta} d\mu.
  \]

- Note that
  \[
  h_{1/2}^2(p, q) \equiv h^2(p, q) = \frac{1}{2} \int \{\sqrt{p} - \sqrt{q}\}^2 d\mu
  \]
  yields the Hellinger distance between $p$ and $q$. By Hölder’s inequality, $h_\beta(p, q) \geq 0$ with equality if and only if $p = q$ a.e. $\mu$. 

Proposition 1.3. Suppose that $\mathcal{P}$ is convex. Then

$$h_{1-\alpha/2}^2(\hat{p}_n, p_0) \leq (\mathbb{P}_n - P_0) \left( \varphi_{\alpha} \left( \frac{\hat{p}_n}{p_0} \right) \right).$$

In particular, when $\alpha = 1$ we have, with $\varphi \equiv \varphi_1$,

$$h^2(\hat{p}_n, p_0) = h_{1/2}^2(\hat{p}_n, p_0) \leq (\mathbb{P}_n - P_0) \left( \varphi \left( \frac{\hat{p}_n}{p_0} \right) \right) = (\mathbb{P}_n - P_0) \left( \frac{2\hat{p}_n}{\hat{p}_n + p_0} \right).$$

Corollary 1.2. Suppose that $\{\varphi(p/p_0) : p \in \mathcal{P}\}$ is a $P_0$–Glivenko-Cantelli class. Then for each $0 < \alpha \leq 1$, $h_{1-\alpha/2}(\hat{p}_n, p_0) \to_{a.s.} 0$.

Proof. Since $\mathcal{P}$ is convex and $\hat{p}_n$ maximizes $\mathbb{P}_n \log p$ over $\mathcal{P}$, it follows that

$$\mathbb{P}_n \log \frac{\hat{p}_n}{(1-t)\hat{p}_n + tp_1} \geq 0$$
for all $0 \leq t \leq 1$ and every $p_1 \in \mathcal{P}$; this holds in particular for $p_1 = p_0$. Note that equality holds if $t = 0$. Differentiation of the left side with respect to $t$ at $t = 0$ yields

$$\mathbb{P}_n \frac{p_1}{\hat{p}_n} \leq 1 \quad \text{for every} \quad p_1 \in \mathcal{P}.$$ 

If $L : (0, \infty) \rightarrow \mathbb{R}$ is increasing and $t \mapsto L(1/t)$ is convex, then Jensen’s inequality yields

$$\mathbb{P}_n L \left( \frac{\hat{p}_n}{p_1} \right) \geq L \left( \frac{1}{\mathbb{P}_n (p_1/\hat{p}_n)} \right) \geq L(1) = \mathbb{P}_n L \left( \frac{p_1}{p_1} \right).$$

Choosing $L = \varphi_\alpha$ and $p_1 = p_0$ in this last inequality and noting that $L(1) = 0$, it follows that

$$0 \leq \mathbb{P}_n \varphi_\alpha(\hat{p}_n/p_0) = (\mathbb{P}_n - P_0) \varphi_\alpha(\hat{p}_n/p_0) + P_0 \varphi_\alpha(\hat{p}_n/p_0); \quad (7)$$

(1988), pages 141 - 143. Now we show that

$$P_0 \varphi_\alpha (p/p_0) = \int \frac{p^\alpha - p_0^\alpha}{p^\alpha + p_0^\alpha} dP_0 \leq - \left( 1 - \int p_0^\beta p^{1-\beta} d\mu \right) \quad (8)$$

Note that this holds if and only if

$$-1 + 2 \int \frac{p^\alpha}{p_0^\alpha + p^\alpha} p_0 d\mu \leq -1 + \int p_0^\beta p^{1-\beta} d\mu,$$

or

$$\int p_0^\beta p^{1-\beta} d\mu \geq 2 \int \frac{p^\alpha}{p_0^\alpha + p^\alpha} p_0 d\mu.$$

But this holds if

$$p_0^\beta p^{1-\beta} \geq 2 \frac{p^\alpha p_0}{p_0^\alpha + p^\alpha}.$$

With $\beta = 1 - \alpha/2$, this becomes

$$\frac{1}{2} (p_0^\alpha + p^\alpha) \geq p_0^{\alpha/2} p^{\alpha/2} = \sqrt{P_0^\alpha p^\alpha},$$
and this holds by the arithmetic mean - geometric mean inequality, $\sqrt{ab} \leq (a + b)/2$. Thus (8) holds. Combining (8) with (7) yields the claim of the proposition.

The corollary follows by noting that $\varphi(t) = (t - 1)/(t + 1) = 2t/(t + 1) - 1$. □
3. More basic inequalities: penalized ML & LS

Penalized ML:

- Suppose that $\mathcal{P}$ is a collection of densities described by a “penalty functional” $I(p)$:

$$\mathcal{P} = \{ p : \mathbb{R} \to [0, \infty) : \int p(x)dx = 1, I^2(p) < \infty \}$$

For example, $I^2(p) = \int (p''(x))^2 dx$.

- Suppose that

$$\hat{p}_n = \arg\max_{p \in \mathcal{P}} (\mathbb{P}_n \log(p) - \lambda_n I^2(p)) ;$$

here $\lambda_n$ is a smoothing parameter.

Basic inequality: (van de Geer, 2000, page 175): For $p_0 \in \mathcal{P}$

$$h^2(\hat{p}_n, p_0) + 4\lambda_n^2 I^2(\hat{p}_n) \leq 16(\mathbb{P}_n - P_0)\frac{1}{2}\log \left( \frac{\hat{p}_n + p_0}{2p_0} \right) + 4\lambda_n^2 I^2(p_0)$$
Least squares regression:

- Suppose that $Y_i = g_0(z_i) + W_i$, where $EW_i = 0$, $Var(W_i) \leq \sigma_0^2$.
- $Q_n = n^{-1} \sum_{i=1}^{n} \delta z_i$, $\|g\|^2_n \equiv n^{-1} \sum_{i=1}^{n} g(z_i)^2$.
- $\|y - g\|^2_n = n^{-1} \sum_{i=1}^{n} (Y_i - g(z_i))^2$.
- $\langle w, g \rangle_n = n^{-1} \sum_{i=1}^{n} W_i g(z_i)$.
- $\hat{g}_n \equiv \arg\min_{g \in G} \|y - g\|^2_n$.


$$\|\hat{g}_n - g_0\|^2_n \leq 2 \langle w, \hat{g}_n - g_0 \rangle_n$$

$$= 2n^{-1} \sum_{i=1}^{n} W_i (\hat{g}_n(z_i) - g_0(z_i)),$$
4. Glivenko-Cantelli Theorems:

Bracketing:

Given two functions \( l \) and \( u \) on \( X \), the bracket \([l, u]\) is the set of all functions \( f \in \mathcal{F} \) with \( l \leq f \leq u \). The functions \( l \) and \( u \) need not belong to \( \mathcal{F} \), but are assumed to have finite norms. An \( \epsilon \)-bracket is a bracket \([l, u]\) with \( \|u - l\| \leq \epsilon \). The bracketing number \( N[\epsilon, \mathcal{F}, \| \cdot \|] \) is the minimum number of \( \epsilon \)-brackets needed to cover \( \mathcal{F} \). The entropy with bracketing is the logarithm of the bracketing number.

**Theorem 1.** Let \( \mathcal{F} \) be a class of measurable functions such that \( N[\epsilon, \mathcal{F}, L_1(P)] < \infty \) for every \( \epsilon > 0 \). Then \( \mathcal{F} \) is \( P \)-Glivenko-Cantelli; that is

\[
\|P_n - P\|_{\mathcal{F}}^* = \left( \sup_{f \in \mathcal{F}} |P_n f - Pf| \right)^* \to \text{a.s.} 0.
\]
Proof. Fix $\epsilon > 0$. Choose finitely many $\epsilon$—brackets $[l_i, u_i]$, $i = 1, \ldots, m = N(\epsilon, \mathcal{F}, L_1(P))$, whose union contains $\mathcal{F}$ and such that $P(u_i - l_i) < \epsilon$ for all $1 \leq i \leq m$. Thus, for every $f \in \mathcal{F}$ there is a bracket $[l_i, u_i]$ such that

$$(\mathbb{P}_n - P)f \leq (\mathbb{P}_n - P)u_i + P(u_i - f) \leq (\mathbb{P}_n - P)u_i + \epsilon.$$ 

Similarly,

$$(P - \mathbb{P}_n)f \leq (P - \mathbb{P}_n)l_i + P(f - l_i) \leq (P - \mathbb{P}_n)l_i + \epsilon.$$ 

It is not hard to see that bracketing condition of Theorem 1 is sufficient but not necessary.

In contrast, our second Glivenko-Cantelli theorem gives conditions which are both necessary and sufficient.
A simple setting in which this theorem applies involves a collection of functions $f = f(\cdot, t)$ indexed or parametrized by $t \in T$, a compact subset of a metric space $(\mathbb{D}, d)$. Here is the basic lemma; it goes back to Wald (1949) and Le Cam (1953).

Lemma 1. Suppose that $\mathcal{F} = \{f(\cdot, t) : t \in T\}$ where the functions $f : \mathcal{X} \times T \mapsto \mathbb{R}$, are continuous in $t$ for $P$– almost all all $x \in \mathcal{X}$. Suppose that $T$ is compact and that the envelope function $F$ defined by $F(x) = \sup_{t \in T} |f(x, t)|$ satisfies $P^*F < \infty$. Then

$$N(\epsilon, \mathcal{F}, L_1(P)) < \infty$$

for every $\epsilon > 0$, and hence $\mathcal{F}$ is $P$–Glivenko-Cantelli.
The qualitative statement of the preceding lemma can be quantified as follows:

**Lemma 2.** Suppose that \( \{f(\cdot, t) : t \in T\} \) is a class of functions satisfying

\[
|f(x, t) - f(x, s)| \leq d(s, t)F(x)
\]

for all \( s, t \in T, \ x \in \mathcal{X} \) for some metric \( d \) on the index set, and a function \( F \) on the sample space \( \mathcal{X} \). Then, for any norm \( \| \cdot \| \),

\[
N[\|2\varepsilon\|F\|, \mathcal{F}, \| \cdot \|) \leq N(\varepsilon, T, d).
\]
For our second Glivenko-Cantelli theorem, we need:

- An **envelope** function $F$ for a class of functions $\mathcal{F}$ is any function satisfying
  \[ |f(x)| \leq F(x) \quad \text{for all } x \in \mathcal{X} \text{ and for all } f \in \mathcal{F}. \]
- A class of functions $\mathcal{F}$ is $L_1(P)$ bounded if $\sup_{f \in \mathcal{F}} P|f| < \infty$. 
Theorem 2.. (Vapnik and Chervonenkis (1981), Pollard (1981), Giné and Zinn (1984)). Let $\mathcal{F}$ be a $P$–measurable class of measurable functions that is $L_1(P)$–bounded. Then $\mathcal{F}$ is $P$–Glivenko-Cantelli if and only if both

(i) $P^*F < \infty$.

(ii) 

$$\lim_{n \to \infty} \frac{E^*\log N(\epsilon, \mathcal{F}_M, L_2(\mathbb{P}_n))}{n} = 0$$

for all $M < \infty$ and $\epsilon > 0$ where $\mathcal{F}_M$ is the class of functions $\{f1\{F \leq M\} : f \in \mathcal{F}\}$. 


For \( n \) points \( x_1, \ldots, x_n \) in \( \mathcal{X} \) and a class of \( \mathcal{C} \) of subsets of \( \mathcal{X} \), set

\[
\Delta_n^\mathcal{C}(x_1, \ldots, x_n) \equiv \# \{ C \cap \{x_1, \ldots, x_n\} : C \in \mathcal{C} \}.
\]

**Corollary.** (Vapnik-Chervonenkis-Steele GC theorem) If \( \mathcal{C} \) is a \( P \)-measurable class of sets, then the following are equivalent:

(i) \( \| P_n - P \|^*_\mathcal{C} \to_{a.s.} 0 \)

(ii) \( n^{-1} E \log \Delta_n^\mathcal{C}(X_1, \ldots, X_n) \to 0 \); where,

The second hypothesis is often verified by applying the theory of VC (or Vapnik-Chervonenkis) classes of sets and functions. Let

\[
m^\mathcal{C}(n) \equiv \max_{x_1, \ldots, x_n} \Delta_n^\mathcal{C}(x_1, \ldots, x_n),
\]

and let

\[
V(\mathcal{C}) \equiv \inf \{ n : m^\mathcal{C}(n) < 2^n \},
\]

\[
S(\mathcal{C}) \equiv \sup \{ n : m^\mathcal{C}(n) = 2^n \}.
\]
Examples:

(1) \( \mathcal{X} = \mathbb{R}, \ C = \{(-\infty, t] : t \in \mathbb{R}\} : S(C) = 1. \)

(2) \( \mathcal{X} = \mathbb{R}, \ C = \{(s, t] : s < t, s, t \in \mathbb{R}\} : S(C) = 2. \)

(3) \( \mathcal{X} = \mathbb{R}^d, \ C = \{(s, t] : s < t, s, t \in \mathbb{R}^d\} : S(C) = 2d. \)

(4) \( \mathcal{X} = \mathbb{R}^d, \ H_{u,c} \equiv \{x \in \mathbb{R}^d : \langle x, u \rangle \leq c\}, \)

\[ C = \{H_{u,c} : u \in \mathbb{R}^d, c \in \mathbb{R}\} : S(C) = d + 1. \]

(5) \( \mathcal{X} = \mathbb{R}^d, \ B_{u,r} \equiv \{x \in \mathbb{R}^d : \|x - u\| \leq r\}; \)

\[ C = \{B_{u,r} : u \in \mathbb{R}^d, r \in \mathbb{R}^+\} : S(C) = d + 1. \]

Definition. The subgraph of \( f : \mathcal{X} \to \mathbb{R} \) is the subset of \( \mathcal{X} \times \mathbb{R} \) given by \( \{(x, t) \in \mathcal{X} \times \mathbb{R} : t < f(x)\} \). A collection of functions \( \mathcal{F} \) from \( \mathcal{X} \) to \( \mathbb{R} \) is called a VC-subgraph class if the collection of subgraphs in \( \mathcal{X} \times \mathbb{R} \) is a VC-class of sets. For a VC-subgraph class \( \mathcal{F} \), let \( V(\mathcal{F}) \equiv V(\text{subgraph}(\mathcal{F})) \).
**Theorem.** For a VC-subgraph class with envelope function $F$ and $r \geq 1$, and for any probability measure $Q$ with $\|F\|_{L_r(Q)} > 0$,

$$N(2\epsilon \|F\|_{Q,r}, F, L_r(Q)) \leq KV(F) \left( \frac{16e}{\epsilon r} \right)^{S(F)}.$$

Here is a specific result for monotone functions on $\mathbb{R}$:

**Theorem.** Let $\mathcal{F}$ be the class of all monotone functions $f : \mathbb{R} \rightarrow [0, 1]$. Then:

(i) (Birman and Solomojak (1967), van de Geer (1991)):

$$\log N(\epsilon, \mathcal{F}, L_r(Q)) \leq \frac{K}{\epsilon}$$

for every probability measure $Q$, every $r \geq 1$, and a constant $K$ depending on $r$ only.

(ii) (via convex hull theory):

$$\sup_Q \log N(\epsilon, \mathcal{F}, L_2(Q)) \leq \frac{K}{\epsilon}$$
Part III: Using the Glivenko-Cantelli theorems: first applications

1. Preservation of Glivenko-Cantelli theorems.
   ▶ Preservation under continuous functions.
   ▶ Preservation under partitions of the sample space.

2. First applications
   ▶ Example 1: current status data
   ▶ Example 2: Mixed case interval censoring
   ▶ Example 3: Completely monotone densities.
1. **Preservation** of Glivenko-Cantelli theorems.

**Theorem 1.** (van der Vaart & W, 2001). Suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_k$ are $P$–Glivenko-Cantelli classes of functions, and that $\varphi : \mathbb{R}^k \to \mathbb{R}$ is continuous. Then $\mathcal{H} \equiv \varphi(\mathcal{F}_1, \ldots, \mathcal{F}_k)$ is $P$–Glivenko-Cantelli provided that it has an integrable envelope function.

**Corollary 1.** (Dudley, 1998). Suppose that $\mathcal{F}$ is a Glivenko-Cantelli class for $P$ with $PF < \infty$, and $g$ is a fixed bounded function ($\|g\|_\infty < \infty$). Then the class of functions $g \cdot \mathcal{F} \equiv \{g \cdot f : f \in \mathcal{F}\}$ is a $P$–Glivenko-Cantelli class.

**Corollary 2.** (Giné and Zinn, 1984). Suppose that $\mathcal{F}$ is a uniformly bounded strong Glivenko-Cantelli class for $P$, and $g \in L_1(P)$ is a fixed function. Then the class of functions $g \cdot \mathcal{F} \equiv \{g \cdot f : f \in \mathcal{F}\}$ is a $P$–Glivenko-Cantelli class.
Theorem 2. (Partitioning of the sample space). Suppose that \( \mathcal{F} \) is a class of functions on \((\mathcal{X}, \mathcal{A}, P)\), and \( \{\mathcal{X}_i\} \) is a partition of \( \mathcal{X} \): \( \bigcup_{i=1}^{\infty} \mathcal{X}_i = \mathcal{X}, \mathcal{X}_i \cap \mathcal{X}_j = \emptyset \) for \( i \neq j \). Suppose that \( \mathcal{F}_j \equiv \{f 1_{\mathcal{X}_j} : f \in \mathcal{F}\} \) is \( P \)-Glivenko-Cantelli for each \( j \), and \( \mathcal{F} \) has an integrable envelope function \( F \). Then \( \mathcal{F} \) is itself \( P \)-Glivenko-Cantelli.
First Applications:

Example 2.1. (Interval censoring, case I). Suppose that $Y \sim F$ on $\mathbb{R}^+$ and $T \sim G$. Here $Y$ is the time of some event of interest, and $T$ is an “observation time”. Unfortunately, we do not observe $(Y, T)$; instead what is observed is $X = (1\{Y \leq T\}, T) \equiv (\Delta, T)$. Our goal is to estimate $F$, the distribution of $Y$. Let $P_0$ be the distribution corresponding to $F_0$, and suppose that $(\Delta_1, T_1), \ldots, (\Delta_n, T_n)$ are i.i.d. as $(\Delta, T)$. Note that the conditional distribution of $\Delta$ given $T$ is simply Bernoulli($F(T)$), and hence the density of $(\Delta, T)$ with respect to the dominating measure $\# \times G$ (here $\#$ denotes counting measure on $\{0, 1\}$) is given by

$$p_F(\delta, t) = F(t)^{\delta}(1 - F(t))^{1-\delta}.$$ 

Note that the sample space in this case is

$$\mathcal{X} = \{(\delta, t) : \delta \in \{0, 1\}, t \in \mathbb{R}^+\} = \{(1, t) : t \in \mathbb{R}^+\} \cup \{(0, t) : t \in \mathbb{R}^+\} =: \mathcal{X}_1 \cup \mathcal{X}_2.$$
Now the class of functions \( \{ p_F : F \text{ a d.f. on } \mathbb{R}^+ \} \) is a universal Glivenko-Cantelli class by an application of GC-preservation Theorem 2, since on \( \mathcal{X}_1 \), \( p_F(1,t) = F(t) \), while on \( \mathcal{X}_2 \), \( p_F(0,t) = 1 - F(t) \) where \( F \) is a distribution \( F \) (and hence bounded and monotone nondecreasing). Furthermore the class of functions \( \{ p_F/p_{F_0} : F \text{ a d.f. on } \mathbb{R}^+ \} \) is \( P_0 \)-Glivenko by an application of GC-preservation Theorem 1: Take

\[
\mathcal{F}_1 = \{ p_F : F \text{ a d.f. on } \mathbb{R}^+ \}, \quad \mathcal{F}_2 = \{ 1/p_{F_0} \},
\]

and \( \varphi(u,v) = uv \). Then both \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are \( P_0 \)-Glivenko-Cantelli classes, \( \varphi \) is continuous, and \( \mathcal{H} = \varphi(\mathcal{F}_1,\mathcal{F}_2) \) has \( P_0 \)-integrable envelope \( 1/p_{F_0} \). Finally, by a further application of GC-preservation Theorem 2 with \( \varphi(u) = (t-1)/(t+1) \) shows that the hypothesis of Corollary 2.1.1 holds: \( \{ \varphi(p_F/p_{F_0}) : F \text{ a d.f. on } \mathbb{R}^+ \} \) is \( P_0 \)-Glivenko-Cantelli. Hence the conclusion of the corollary holds: we conclude that

\[
h^2(p_{\hat{F}_n},p_{F_0}) \to_{a.s.} 0 \quad \text{as} \quad n \to \infty.
\]
Now note that \( h^2(p, p_0) \geq d_{TV}^2(p, p_0)/2 \) and we compute

\[
d_{TV}(p_{\hat{F}_n}, p_{F_0}) = \int |\hat{F}_n(t) - F_0(t)|dG(t) \\
+ \int |1 - \hat{F}_n(t) - (1 - F_0(t))|dG(t) \\
= 2 \int |\hat{F}_n(t) - F_0(t)|dG(t),
\]

so we conclude that

\[
\int |\hat{F}_n(t) - F_0(t)|dG(t) \to_{a.s.} 0
\]
as \( n \to \infty \). Since \( \hat{F}_n \) and \( F_0 \) are bounded (by one), we can also conclude that

\[
\int |\hat{F}_n(t) - F_0(t)|^r dG(t) \to_{a.s.} 0
\]
for each \( r \geq 1 \), in particular for \( r = 2 \).
Example 2. (Mixed case interval censoring)

Suppose that:

- \( Y \sim F \) on \( R^+ = [0, \infty) \).

- Observe:
  - \( T_K = (T_{K,1}, \ldots, T_{K,K}) \) where \( K \), the number of times is itself random.
  - The interval \( (T_{K,j-1}, T_{K,j}] \) into which \( Y \) falls (with \( T_{K,0} \equiv 0, T_{K,K+1} \equiv \infty \)).
  - Here \( K \in \{1, 2, \ldots\} \), and \( \underline{T} = \{T_{k,j}, j = 1, \ldots, k, k = 1, 2, \ldots\} \).
  - \( Y \) and \((K, \underline{T})\) are independent.

- \( X \equiv (\Delta_K, T_K, K) \), with a possible value \( x = (\delta_k, t_k, k) \), where \( \Delta_k = (\Delta_{k,1}, \ldots, \Delta_{k,k}) \) with \( \Delta_{k,j} = 1(T_{k,j-1}, T_{k,j}](Y), j = 1, 2, \ldots, k + 1. \)
• Suppose we observe $n$ i.i.d. copies of $X_1, X_2, \ldots, X_n$, where $X_i = (\Delta_{K(i)}, T_{K(i)}^{(i)}, K(i)), \ i = 1, 2, \ldots, n$. Here $(Y(i), T(i), K(i)), \ i = 1, 2, \ldots$ are the underlying i.i.d. copies of $(Y, T, K)$.

note that conditionally on $K$ and $T_K$, the vector $\Delta_K$ has a multinomial distribution:

$$(\Delta_K | K, T_K) \sim \text{Multinomial}_{K+1}(1, \Delta F_K)$$

where

$$\Delta F_K = (F(T_{K,1}), F(T_{K,2}) - F(T_{K,1}), \ldots, 1 - F(T_{K,K})).$$
Suppose for the moment that the distribution $G_k$ of $(T_K|K=\kappa)$ has density $g_k$ and $p_k \equiv P(K=\kappa)$. Then a density of $X$ is given by

\[ p_F(x) \equiv p_F(\delta, t_k, k) \]

\[ = \prod_{j=1}^{k+1} (F(t_{k,j}) - F(t_{k,j-1})) \delta_{k,j} g_k(t)p_k \]

where $t_{k,0} \equiv 0$, $t_{k,k+1} \equiv \infty$. In general,

\[ p_F(x) \equiv p_F(\delta, t_k, k) \]

\[ = \prod_{j=1}^{k+1} (F(t_{k,j}) - F(t_{k,j-1})) \delta_{k,j} \]

\[ = \sum_{j=1}^{k+1} \delta_{k,j} (F(t_{k,j}) - F(t_{k,j-1})) \]

is a density of $X$ with respect to the dominating measure $\nu$ where $\nu$ is determined by the joint distribution of $(K,T)$, and it is this
version of the density of $X$ with which we will work throughout the rest of the example. Thus the log-likelihood function for $F$ of $X_1, \ldots, X_n$ is given by

$$\frac{1}{n} \ln (F|X) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{K^{(i)}+1} \Delta_{K,j}^{(i)} \log \left( F(T_{K(i),j}^{(i)}) - F(T_{K(i),j-1}^{(i)}) \right)$$

$$= P_n m_F$$

where

$$m_F(X) = \sum_{j=1}^{K+1} \Delta_{K,j} \log \left( F(T_{K,j} - F(T_{K,j-1}) \right)$$

$$\equiv \sum_{j=1}^{K+1} \Delta_{K,j} \log \left( \Delta F_{K,j} \right)$$

and where we have ignored the terms not involving $F$. We also
note that
\[ Pm_F(X) = P \left( \sum_{j=1}^{K+1} \Delta F_{0,K,j} \log \left( \Delta F_{K,j} \right) \right). \]

The (Nonparametric) Maximum Likelihood Estimator (MLE)
\[ \hat{F}_n = \arg\max_{F} \mathbb{P}_n \ell_n(F). \]
\( \hat{F}_n \) can be calculated via the iterative convex minorant algorithm proposed in Groeneboom and Wellner (1992) for case 2 interval censored data.
By Proposition 1 with $\alpha = 1$ and $\varphi \equiv \varphi_1$ as before, it follows that

$$h^2(p_{\hat{F}_n}, p_{F_0}) \leq (\mathbb{P}_n - P_0) \left( \varphi \left( p_{\hat{F}_n} / p_{F_0} \right) \right)$$

where $\varphi$ is bounded and continuous from $\mathbb{R}$ to $\mathbb{R}$. Now the collection of functions

$$\mathcal{G} \equiv \{ p_F : F \in \mathcal{F} \}$$

is easily seen to be a Glivenko-Cantelli class of functions: this can be seen by first applying the GC-preservation theorem Theorem 1 to the collections $\mathcal{G}_k$, $k = 1, 2, \ldots$ obtained from $\mathcal{G}$ by restricting to the sets $K = k$. Then for fixed $k$, the collections $\mathcal{G}_k = \{ p_F(\delta, t_k, k) : F \in \mathcal{F} \}$ are $P_0$-Glivenko-Cantelli classes since $\mathcal{F}$ is a uniform Glivenko-Cantelli class, and since the functions $p_F$ are continuous transformations of the classes of functions $x \to \delta_{k,j}$ and $x \to F(t_{k,j})$ for $j = 1, \ldots, k + 1$, and hence $\mathcal{G}$ is $P$-Glivenko-Cantelli by van de Geer’s bracketing entropy bound for monotone
functions. Note that single function $p_{F_0}$ is trivially $P_0-$ Glivenko-Cantelli since it is uniformly bounded, and the single function $(1/p_{F_0})$ is also $P_0-$ GC since $P_0(1/p_{F_0}) < \infty$. Thus by the Glivenko-Cantelli preservation Theorem 1 with $g = (1/p_{F_0})$ and $\mathcal{F} = \mathcal{G} = \{p_F : F \in \mathcal{F}\}$, it follows that $\mathcal{G}' \equiv \{p_F/p_{F_0} : F \in \mathcal{F}\}$. Is $P_0-$Glivenko-Cantelli. Finally another application of preservation of the Glivenko-Cantelli property by continuous maps shows that the collection

$$\mathcal{H} \equiv \{\varphi(p_F/p_{F_0}) : F \in \mathcal{F}\}$$

is also $P_0$-Glivenko-Cantelli. When combined with Corollary 1.1, we find:

**Theorem.** The NPMLE $\hat{F}_n$ satisfies

$$h(\hat{F}_n, p_{F_0}) \to_{a.s.} 0.$$  

To relate this result to a result of Schick and Yu (2000), it remains only to understand the relationship between their $L_1(\mu)$
and the Hellinger metric \( h \) between \( p_F \) and \( p_{F_0} \). Let \( \mathcal{B} \) denote the collection of Borel sets in \( R \). On \( \mathcal{B} \) we define measures \( \mu \) and \( \tilde{\mu} \), as follows: For \( B \in \mathcal{B} \),

\[
\mu(B) = \sum_{k=1}^{\infty} P(K = k) \sum_{j=1}^{k} P(T_{k,j} \in B | K = k),
\]

(10)

and

\[
\tilde{\mu}(B) = \sum_{k=1}^{\infty} P(K = k) \frac{1}{k} \sum_{j=1}^{k} P(T_{k,j} \in B | K = k).
\]

(11)

Let \( d \) be the \( L_1(\mu) \) metric on the class \( \mathcal{F} \); thus for \( F_1, F_2 \in \mathcal{F} \),

\[
d(F_1, F_2) = \int |F_1(t) - F_2(t)| d\mu(t).
\]

The measure \( \mu \) was introduced by Schick and Yu (2000); note that \( \mu \) is a finite measure if \( E(K) < \infty \). Note that \( d(F_1, F_2) \) can
also be written in terms of an expectation as:

$$d(F_1, F_2) = E_{(K,T)} \left[ \sum_{j=1}^{K+1} |F_1(T_{K,j}) - F_2(T_{K,j})| \right].$$ \hspace{1cm} (12)

As Schick and Yu (2000) observed, consistency of the NPMLE $\hat{F}_n$ in $L_1(\mu)$ holds under virtually no further hypotheses.

**Theorem.** (Schick and Yu). Suppose that $E(K) < \infty$. Then $d(\hat{F}_n, F_0) \rightarrow_{a.s.} 0$.

**Proof.** We have shown that this follows from the Hellinger consistency proved above and the following lemma; see van der Vaart and Wellner (2000).

**Lemma.**

$$\frac{1}{2} \left\{ \int |\hat{F}_n - F_0| d\tilde{\mu} \right\}^2 \leq h^2(p_{\hat{F}_n}, p_{F_0}).$$
Example 3. (Completely monotone densities:)

Suppose that $\mathcal{P} = \{P_G : G \text{ a d.f. on } \mathbb{R}\}$ where the measures $P_G$ are scale mixtures of exponential distributions with mixing distribution $G$:

$$p_G(x) = \int_{0}^{\infty} ye^{-yx} dG(y).$$

We first show that the map $G \mapsto p_G(x)$ is continuous with respect to the topology of vague convergence for distributions $G$. This follows easily since kernels for our mixing family are bounded, continuous, and satisfy $ye^{-xy} \to 0$ as $y \to \infty$ for every $x > 0$. Since vague convergence of distribution functions implies that integrals of bounded continuous functions vanishing at infinity converge, it follows that $p(x; G)$ is continuous with respect to the vague topology for every $x > 0$.

This implies, moreover, that the family $\mathcal{F} = \{p_G/(p_G + p_0) : G \text{ is a d.f. on } \mathbb{R}\}$ is pointwise, for a.e. $x$, continuous in $G$. 

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with respect to the vague topology. Since the family of sub-
distribution functions \( G \) on \( R \) is compact for (a metric for) the
vague topology (see e.g. Bauer (1972), page 241), and the
family of functions \( F \) is uniformly bounded by 1, we conclude
from the basic bracketing lemma (Wald and LeCam) that
\( N_{[]} (\epsilon, F, L_1(P)) < \infty \) for every \( \epsilon > 0 \). Thus it follows from
Corollary 1.1 that the MLE \( \hat{G}_n \) of \( G_0 \) satisfies

\[
h(p_{\hat{G}_n}, p_{G_0}) \to a.s. 0.
\]

By uniqueness of Laplace transforms, this implies that \( \hat{G}_n \)
converges weakly to \( G_0 \) with probability 1. This method of proof
is due to Pfanzagl (1988); in this case we recover a result of
Jewell (1982). See also Van de Geer (1999), Example 4.2.4,
page 54.
Xièxiè!