Estimation and Testing
with Current Status Data

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• joint work with Moulinath Banerjee, University of Michigan

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Outline

• Introduction: current status data
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• Estimation of $F$, unconstrained (old, Ayer et al. 1956)
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• Confidence intervals for $F(t_0)$
• Further problems
1. Introduction: current status data

- \( X \sim F, \ Y \sim G, \ X, Y \text{ independent} \)
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- $X \sim F$, $Y \sim G$, $X, Y$ independent
- We observe $(Y, 1\{X \leq Y\}) \equiv (Y, \Delta)$ with density

$$p_F(y, \delta) = F(y)^\delta (1 - F(y))^{1-\delta} g(y)$$
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- \( X \sim F, Y \sim G, \; X, Y \) independent
- We observe \((Y, 1\{X \leq Y\}) \equiv (Y, \Delta)\) with density
  \[
p_F(y, \delta) = F(y)^\delta (1 - F(y))^{1-\delta} g(y)
\]
- Suppose that \((Y_i, \Delta_i)\) are i.i.d. as \((Y, \Delta)\).
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- $X \sim F, \ Y \sim G, \ \ X, Y$ independent
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$$p_F(y, \delta) = F(y)^\delta (1 - F(y))^{1-\delta} g(y)$$

- Suppose that $(Y_i, \Delta_i)$ are i.i.d. as $(Y, \Delta)$.
- Likelihood:

$$L_n(F) = \prod_{i=1}^{n} F(Y_i)^{\Delta_i} (1 - F(Y_i))^{1-\Delta_i}$$
Likelihood ratio test of $H : F(t_0) = \theta_0$
- Likelihood ratio test of \( H : F(t_0) = \theta_0 \)
- The likelihood ratio statistic:

\[
\lambda_n = \frac{\sup_F L_n(F)}{\sup_{F : F(t_0) = \theta_0} L_n(F)} = \frac{L_n(\hat{F}_n)}{L_n(\hat{F}_n^0)}.
\]
2. Estimation of $F$: Nonparametric MLE

- MLE (Unconstrained) \[ \hat{F}_n(t) = \arg\max_F L_n(F) \]
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- MLE (Unconstrained) \[ \hat{F}_n(t) = \arg\max_F L_n(F) \]
- Another description: define

\[
\begin{align*}
G_n(t) &= \frac{1}{n} \sum_{i=1}^{n} 1_{[Y_i \leq t]}, \\
\nV_n(t) &= \frac{1}{n} \sum_{i=1}^{n} \Delta_i 1_{[Y_i \leq t]}.
\end{align*}
\]

Note that

\[
G_n(t) \rightarrow_{a.s.} G(t), \quad V_n(t) \rightarrow_{a.s.} \int_0^t F(y) dG(y) \equiv V(t).
\]
2. Estimation of $F$: Nonparametric MLE

- MLE (Unconstrained) \[ \hat{F}_n(t) = \arg\max_F L_n(F) \]
- Another description: define

\[ G_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1_{[Y_i \leq t]}, \quad \nabla_n(t) = \frac{1}{n} \sum_{i=1}^{n} \Delta_i 1_{[Y_i \leq t]} . \]

Note that

\[ G_n(t) \to_{a.s} G(t), \quad \nabla_n(t) \to_{a.s} \int_0^t F(y) dG(y) \equiv V(t) . \]

- Thus

\[ \frac{dV}{dG}(t) = F(t) . \]
Partial sum diagram: Let $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$. The partial sum diagram $\mathcal{P} = \{ P_i \}$ is given by

$$P_i = (G_n(Y_{(i)}), V_n(Y_{(i)})), \quad i = 1, \ldots, n.$$
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$$i = 1, \ldots, n.$$

• The Nonparametric MLE $\hat{F}_n$ of $F$ is:

$$\hat{F}_n(Y_{(i)}) = \text{left derivative of the Greatest Convex Minorant of } \mathcal{P} \text{ at } Y_{(i)}.$$
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• Greatest Convex Minorant = GCM
Cumulative Sum Diagram: Example $n = 5$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta(i)$</th>
<th>$Y(i)$</th>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.4</td>
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$\hat{\nabla}_n(Y(i))$
Cumulative Sum Diagram: Example $n = 5$

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$nnV_n(Y_{(i)})$
Example continued, \( n = 5 \)

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\[ \hat{F}_n \]

\[ y \]

1

\[ 0.5 \]

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Example continued, $n = 5$

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\[
\hat{F}_n
\]
$F = \text{Exponential}(1); G = \text{Uniform}(0, 3); n = 30$
3. The constrained MLE $\hat{F}_0^n$. Recipe:

- Break $\mathcal{P}$ into $\mathcal{P}_L$ and $\mathcal{P}_R$ where

$$\mathcal{P}_L = \{P_i : Y(i) \leq t_0\}, \quad \mathcal{P}_R = \{P_i : Y(i) > t_0\}.$$
3. The constrained MLE $\hat{F}_n^0$. Recipe:

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- Form the GCM’s of $\mathcal{P}_L$ and $\mathcal{P}_R$, say $\tilde{\mathcal{V}}^L_n$ and $\tilde{\mathcal{V}}^R_n$. 
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$$\mathcal{P}_{L} = \{P_{i} : Y_{(i)} \leq t_{0}\}, \quad \mathcal{P}_{R} = \{P_{i} : Y_{(i)} > t_{0}\}.$$

- Form the GCM’s of $\mathcal{P}_{L}$ and $\mathcal{P}_{R}$, say $\tilde{V}_{L}^{n}$ and $\tilde{V}_{R}^{n}$.

- If the slope of $\tilde{V}_{L}^{n}$ exceeds $\theta_{0}$, replace it by $\theta_{0}$; if the slope of $\tilde{V}_{R}^{n}$ drops below $\theta_{0}$, replace it by $\theta_{0}$.
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- The resulting (truncated or constrained) slope process yields the constrained MLE $\hat{F}_n^0$. 

Estimation and Testing with Current Status Data – p. 13/44
$F =$Exponential(1); $G =$Uniform(0, 3); $n = 30$; $F(t_0) = 2/3$;
$t_0 = -\log(1 - 2/3) = 1.09861...$
\[ F = \text{Exponential}(1); \quad G = \text{Uniform}(0, 3); \quad n = 30; \quad F(t_0) = \frac{2}{3}; \]
\[ t_0 = -\log(1 - \frac{2}{3}) = 1.09861... \]
4. The likelihood ratio test of $H : F(t_0) = \theta_0$

- Likelihood ratio statistic:

$$\lambda_n = \frac{\sup_F L_n(F)}{\sup_{F : F(t_0) = \theta_0} L_n(F)} = \frac{L_n(\hat{F}_n)}{L_n(\hat{F}_0^n)}.$$
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$$\lambda_n = \frac{\sup_F L_n(F)}{\sup_{F : F(t_0) = \theta_0} L_n(F')} = \frac{L_n(\hat{F}_n)}{L_n(\hat{F}_0^0)}.$$

- How big is “too big”? 

Estimation and Testing with Current Status Data – p. 16/44
4. The likelihood ratio test of $H : F(t_0) = \theta_0$

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- How big is “too big”?
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$$2 \log \lambda_n \to_d \text{something}$$
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\]

- How big is “too big”?
- When $H : F(t_0) = \theta_0$ holds, does $2 \log \lambda_n \rightarrow_d$ something?

- Answer: Yes! Banerjee and Wellner (2001)
5. How big is “too big”? The limiting Gaussian problem

- Suppose that we observe \( \{X(t) : t \in R\} \) where

\[
X(t) = F(t) + \sigma W(t)
\]

- \( F(t) = \int_{-\infty}^{t} f(s)ds, \)
- \( f \) monotone non-decreasing, and
- \( W \) is standard (two-sided) Brownian motion.
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• Suppose that we want to estimate the monotone function \( f \). Equivalently

\[
dX(t) = f(t)dt + \sigma dW(t).
\]
5. How big is “too big”? The limiting Gaussian problem

- Suppose that we observe \( \{X(t) : t \in \mathbb{R}\} \) where
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  - \( F(t) = \int_{-\infty}^{t} f(s) ds \),
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- Suppose that we want to estimate the monotone function \( f \).
  Equivalently
  \[
  dX(t) = f(t) dt + \sigma dW(t).
  \]

- The “canonical monotone function” is a linear one, and we can change \( \sigma \) to 1 by virtue of scaling arguments so the “canonical” version of the problem is as follows:
  \[
  dX(t) = 2t dt + dW(t),
  \]
• “estimate” $2t$ when $\{X(t) : t \in R\}$, is observed. Thus

$$X(t) = t^2 + W(t).$$

Unconstrained “Estimator”: Slope of GCM of $X(t)$. Call this process of slopes of the GCM $S$. 
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Unconstrained “Estimator”: Slope of GCM of \( X(t) \). Call this process of slopes of the GCM $\mathcal{S}$. 
$W(t) + t^2$

The diagram shows the samplepath $W(t)$ and an unconstrained path. The $t$-axis represents time, and the $W(t)$ axis represents the value of $W(t)$ at different times $t$. The samplepath is represented by the black line, while the unconstrained path is represented by the red line.
• What is the “canonical constrained problem”? 
• What is the “canonical constrained problem”?
• Estimate the monotone function $f(t) = 2t$ subject to the constraint that $f(0) = 0$ when $\{X(t) : t \in R\}$ is observed.
What is the "constrained estimator"?
What is the “constrained estimator”?

Recipe:

- Break \( \{X(t) : t \in R\} \) into \( X^L \equiv \{X(t) : t < 0\} \) and \( X^R \equiv \{X(t) : t \geq 0\} \).
- Form the GCM’s of \( X^L \) and \( X^R \) say \( Y^L \) and \( Y^R \).
- If the slope of \( Y^L \) exceeds 0, replace it by 0; if the slope of \( Y^R \) drops below 0, replace it by 0.
- The resulting (truncated or constrained) slope process \( S^0 \) is the constrained MLE of \( f(t) = 2t \) in the Gaussian problem.
Estimation and Testing with Current Status Data – p. 25/44
Likelihood ratio test statistic in the Gaussian problem?

- Suppose $\{X(t) : t \in [-c, c]\}$ is given by

\[ dX(t) = f(t)dt + dW(t) \]

so

\[ X(t) = F(t) + W(t). \]
Likelihood ratio test statistic in the Gaussian problem?

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\[
dX(t) = f(t)\,dt + dW(t)
\]

so

\[
X(t) = F(t) + W(t).
\]

• Radon-Nikodym derivative (drifted process relative to zero drift):

\[
\frac{dP_f}{dP_0} = \exp \left( \int_{-c}^{c} f\,dX - \frac{1}{2} \int_{-c}^{c} f^2(t)\,dt \right).
\]
Likelihood ratio test statistic in the Gaussian problem?

• Suppose \( \{X(t) : t \in [-c, c]\} \) is given by

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\]

so

\[
X(t) = F(t) + W(t).
\]

• Radon-Nikodym derivative (drifted process relative to zero drift):

\[
\frac{dP_f}{dP_0} = \exp\left(\int_{-c}^{c} f dX - \frac{1}{2} \int_{-c}^{c} f^2(t)dt\right).
\]

• \( \mathcal{F}(c, K) = \{ \text{monotone functions } f : [-c, c] \to \mathbb{R}, \|f\|_c \leq K \} \)

\( \mathcal{F}_0(c, K) = \{ f \in \mathcal{F}(c, K) : f(0) = 0 \} \)
• Then

\[
2 \log \lambda_c = 2 \log \left( \frac{\sup_{f \in \mathcal{F}(c, K)} \frac{dP_f}{dP_0}}{\sup_{f \in \mathcal{F}_0(c, K)} \frac{dP_f}{dP_0}} \right) = 2 \log \left( \frac{dP_\hat{f}/dP_0}{dP_{\hat{f}_0}/dP_0} \right)
\]

\[
= 2 \left\{ \int_c^c \hat{f}_c dX - \frac{1}{2} \int_{-c}^c \hat{f}_c^2(t) dt \right. \\
- \int_c^c \hat{f}_{c,0} dX + \frac{1}{2} \int_{-c}^c \hat{f}_{c,0}^2(t) dt \right\}
\]

\[
= 2 \int_{-c}^c (\hat{f}_c - \hat{f}_{c,0}) dX - \int_{-c}^c \{\hat{f}_c^2(t) - \hat{f}_{c,0}^2(t)\} dt .
\]
• Taking the limit as $c \to \infty$ with $K = K_c = 5c$, this yields

$$2 \log \lambda = 2 \int_D (\hat{f} - \hat{f}_0) dX - \int_D \{\hat{f}^2(t) - \hat{f}_0^2(t)\} dt$$
• Taking the limit as $c \to \infty$ with $K = K_c = 5c$, this yields

\[2 \log \lambda = 2 \int_D (\hat{f} - \hat{f}_0) dX - \int_D \{\hat{f}^2(t) - \hat{f}_0^2(t)\} dt\]

• From the characterizations of $\hat{f}$ and $\hat{f}_0$:

\[
\int_{\mathbb{R}} (X - \hat{F}) d\hat{f} = 0, \quad \int_{\mathbb{R}} (X - \hat{F}_0) d\hat{f}_0 = 0.
\]
Integration by parts:

\[ \int_{\mathbb{R}} (\hat{f} - \hat{f}_0) \, dX = \int_D (\hat{f} - \hat{f}_0) \, dX = -\int_D X \, d(\hat{f} - \hat{f}_0) \]

\[ = -\int_D \hat{F} \, d\hat{f} + \int_D \hat{F}_0 \, d\hat{f}_0 \]

\[ = \int_D \hat{f} \, d\hat{F} - \int_D \hat{f}_0 \, d\hat{F}_0 \]

\[ = \int_D \{ \hat{f}^2(t) - \hat{f}_0^2(t) \} \, dt. \]
• Integration by parts:

\[
\int_{\mathbb{R}} (\hat{f} - \hat{f}_0) dX = \int_D (\hat{f} - \hat{f}_0) dX = - \int_D X d(\hat{f} - \hat{f}_0)
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\]

• Likelihood ratio statistic becomes:

\[
2 \log \lambda = \int_D \{\hat{f}^2(t) - \hat{f}_0^2(t)\} dt.
\]
6. Limit distribution, LR statistic under $H$

- Limit distributions for $\hat{F}_n$ and $\hat{F}_n^0$. Set

$$G_n^{loc}(t, h) = n^{1/3} (G_n(t + n^{-1/3}h) - G_n(t))$$

$$V_n^{loc}(t, h) = n^{1/3} \left\{ n^{1/3} (V_n(t + n^{-1/3}h) - V_n(t)) - G_n^{loc}(t, h)F(t) \right\}. $$
6. Limit distribution, LR statistic under $H$

- Limit distributions for $\hat{F}_n$ and $\hat{F}_0^0$. Set

$$G_{loc}^n(t, h) = n^{1/3}(G_n(t + n^{-1/3}h) - G_n(t))$$
$$V_{loc}^n(t, h) = n^{1/3} \left\{ n^{1/3}(V_n(t + n^{-1/3}h) - V_n(t)) - G_{loc}^n(t, h)F(t) \right\}.$$ 

- Theorem 1. If $g(t_0) = G'(t_0)$ and $f(t_0) = F'(t_0)$ exist, then:
  A. $G_{loc}^n(t_0, h) \rightarrow_p g(t_0)h.$
  B. $V_{loc}^n(t_0, h) \Rightarrow aW(h) + bh^2$ where 
  
  $$a = \sqrt{F(t_0)(1 - F(t_0))g(t_0)}, \quad b = f(t_0)g(t_0)/2,$$ 
  and $W$ is a two-sided Brownian motion starting from 0.
• Now define

\[
\begin{align*}
Z_n(h) &= n^{1/3} (\hat{F}_n(t_0 + hn^{-1/3}) - F(t_0)), \\
Z^0_n(h) &= n^{1/3} (\hat{F}^0_n(t_0 + hn^{-1/3}) - F(t_0)).
\end{align*}
\]
• Now define

\[ \mathbb{Z}_n(h) = n^{1/3}(\hat{F}_n(t_0 + hn^{-1/3}) - F(t_0)), \]
\[ \mathbb{Z}_0^n(h) = n^{1/3}(\hat{F}_0^n(t_0 + hn^{-1/3}) - F(t_0)). \]

• Theorem 2. If the hypotheses of Theorem 1 hold with \( f(t_0) > 0, g(t_0) > 0, \) and \( F(t_0) = \theta_0, \) then

\[ (\mathbb{Z}_n(h), \mathbb{Z}_0^n(h)) \Rightarrow (S_{a,b}(h), S_{0,a,b}(h))/g(t_0) \]

where \( S_{a,b} \) and \( S_{a,b}^0 \) are the constrained and unconstrained slope processes corresponding to \( X_{a,b}(h) = aW(h) + bh^2. \)
• Limit distribution for $2 \log \lambda_n$
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• **Theorem 3.** (Banerjee and Wellner, 2001). Suppose that $F$ and $G$ have densities $f$ and $g$ which are strictly positive and continuous in a neighborhood in a neighborhood of $t_0$. Suppose that $F(t_0) = \theta_0$. Then

$$2 \log \lambda_n \xrightarrow{d} \frac{1}{g(t_0)a^2} \int ((S_{a,b}(z))^2 - (S_{a,b}^{0}(z))^2) \, dz$$

$$\overset{d}{=} \int \{(S(z))^2 - (S_{0}(z))^2\} \, dz \equiv \mathbb{D},$$

and the distribution of $\mathbb{D}$ is **universal** (free of parameters).
Estimation and Testing with Current Status Data – p. 33/44
7. Confidence intervals for $F(t_0)$

- Wald-type intervals:

$$Z_n(0) = n^{1/3}(\hat{F}_n(t_0) - F(t_0)) \rightarrow_d S_{a,b}(0)/g(t_0)$$

$$\equiv d \left\{ \frac{F(t_0)(1 - F(t_0))f(t_0)}{2g(t_0)} \right\}^{1/3} \equiv C(F, f, g)S(0)$$

where $S(0) \equiv 2\mathbb{Z} \equiv 2\text{argmin}(W(h) + h^2)$, $S(0) \equiv S_{1,1}(0)$. 

Estimation and Testing with Current Status Data – p. 35/44
7. Confidence intervals for $F(t_0)$

- Wald-type intervals:

$$Z_n(0) = n^{1/3}(\hat{F}_n(t_0) - F(t_0)) \rightarrow_d S_{a,b}(0)/g(t_0)$$

$$= \frac{d}{2g(t_0)} \left\{ \frac{F(t_0)(1-F(t_0))f(t_0)}{2g(t_0)} \right\}^{1/3} S(0)$$

$$\equiv C(F,f,g)S(0)$$

where $S(0) \overset{d}{=} 2Z \equiv 2\arg\min(W(h) + h^2)$, $S(0) \equiv S_{1,1}(0)$.

- Wald - interval:

$$\hat{F}_n(t_0) \pm n^{-1/3}C(\hat{F}_n, \hat{f}_n, \hat{g}_n) t_\alpha$$

where $\hat{f}_n$ and $\hat{g}_n$ are estimates of $f$ and $g$ (at $t_0$), and $t_{\alpha/2}$ satisfies

$$P(2Z > t_{\alpha/2}) = \alpha/2.$$
Problem: this involves smoothing to get estimators $\hat{f}_n$ and $\hat{g}_n$.
• Confidence intervals from the LR test
• Confidence intervals from the LR test
• Invert the test:

\[ \{ \theta : 2 \log \lambda_n(\theta) \leq d_\alpha \} . \]

where \( P(\mathbb{D} \leq d_\alpha) = 1 - \alpha \)
• Confidence intervals from the LR test

• Invert the test:

$$\{ \theta : 2 \log \lambda_n(\theta) \leq d_\alpha \}.$$ 

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• Advantage: no smoothing needed!
• Confidence intervals from the LR test
• Invert the test:
  \[ \{ \theta : 2 \log \lambda_n(\theta) \leq d_\alpha \} . \]
  where \( P(D \leq d_\alpha) = 1 - \alpha \)
• Advantage: no smoothing needed!
• Tradeoff: need to compute constrained estimator(s) \( \hat{F}_n^0 \) of \( F \) and \( \lambda_n(\theta) \) for many different values of the constraint \( \theta \).
Austrian rubella (measles) data, from Niels Keiding

- 230 Austrian males older than 3 months
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  - no vaccination of males
  - males represent an unvaccinated population
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Estimation and Testing with Current Status Data – p. 41/44
Keiding's LRT based Estimation and Testing with Current Status Data – p. 42/44
8. Further problems

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Y_k \equiv D_1 + \cdots + D_k
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where $D_1, \ldots, D_k$ are independent with $D_j \overset{d}{=} D$)
- Confidence bands for the whole monotone function $F$? (No LR bands yet. Bands based on multi-scale methods: Dümbgen (1999))
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• Does the same limit $\mathbb{D}$ arise as the limit distribution for the likelihood ratio test for a large class of such problems involving monotone functions? (Yes Banerjee (2005))

• Similar methods for more complicated models, e.g. competing risks with current status data as in Groeneboom, Maathuis, Wellner (2006a,b)?

• Confidence intervals (and bands?) for estimating a concave distribution function $F$?