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Outline

- Introduction: shape constraints, nonparametric estimation and testing
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• Problems 1-4 from Gothenburg meeting: rearrangements versus maximum likelihood
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• Problems 5-6 from Gothenburg meeting: how big is the Grenander estimator at zero
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- Introduction: shape constraints, nonparametric estimation and testing
- Problems 1-4 from Gothenburg meeting: rearrangements versus maximum likelihood
- Problems 5-6 from Gothenburg meeting: how big is the Grenander estimator at zero
- Four more problems involving shape constraints ... very briefly
1. Introduction: shape constraints

Types of shape restrictions for functions on $\mathbb{R}$:

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Estimation and Testing with Shape Constraints – p. 5/30
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• Confidence sets?
  ◦ Assuming shape constraint?
  ◦ Testing to see if a shape constraint is true?
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Monotone rearrangements estimator versus maximum likelihood?

Continuous setting
$X_1, \ldots, X_n$ i.i.d. with density $f$ on $[0, \infty)$ where $f \downarrow 0$.

The Maximum Likelihood Estimator is

$$\hat{f}_n = \arg\max_{f \in \mathcal{M}_1} \left\{ \sum_{i=1}^{n} \log f(X_i) \right\} = \text{the MLE}$$

$$= \text{Grenander estimator of } f.$$
From Grenander (1956), the MLE is characterized by the Fenchel conditions:

\[ \mathbb{F}_n(x) \leq \hat{F}_n(x) \equiv \int_0^x \hat{f}_n(t) \, dt \quad \text{for all } x \in [0, \infty) , \quad \text{and} \]

\[ \mathbb{F}_n(x) = \hat{F}_n(x) \quad \text{if and only if} \quad \hat{f}_n(x-) > \hat{f}_n(x+). \]

The geometric interpretation of these two conditions is

\[ \hat{f}_n(x) = \left\{ \begin{array}{l} \text{the left-derivative of the slope at } x \text{ of the} \\ \text{least concave majorant } \hat{F}_n \text{ of } \mathbb{F}_n \end{array} \right\} \equiv \partial \mathcal{I}_1(\mathbb{F}_n) \]
Monotone rearrangement estimator

- Monotone rearrangement, continuous case: $\hat{f}^{\text{rearr}} \equiv R(\tilde{f}_n)$

where

$$Z_f(s) = \lambda\{x : f(x) \geq s\}, \quad R(f)(x) = Z_f^{-1}(x).$$
Monotone rearrangement estimator

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Z_f(s) = \lambda \{ x : f(x) \geq s \}, \quad R(f)(x) = Z_f^{-1}(x).
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• Monotone rearrangement, discrete case: \( \hat{p}_n^{\text{rearr}} \equiv R(\tilde{p}_n) \)
where
\[
Z_p(s) = \# \{ i \in \mathbb{N}^+ : p(i) \geq s \}, \quad R(p)(i) = Z_p^{-1}(i).
\]
Estimation and Testing with Shape Constraints – p. 16/30
\{p_x : x \in \mathbb{N}\}, a non-increasing mass function on \mathbb{N}
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• \( \hat{p}_{n,x} \equiv n^{-1} \# \{ i \leq n : X_i = x \} \) for \( x \in \mathbb{N} \).
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• $\hat{p}_{n,x} \equiv n^{-1}\#\{i \leq n : X_i = x\}$ for $x \in \mathbb{N}$.

• $Y_{n,x} \equiv \sqrt{n}(\hat{p}_{n,x} - p_x)$ for $x \in \mathbb{N}$.
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• \( Y_x \) a Gaussian process on \( \mathbb{N} \) with \( \mathbb{E}Y_x = 0 \),

\[
\text{Cov}(Y_x, Y_{x'}) = p_x \delta_{x,x'} - p_x p_{x'}.
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• Define processes \( Y^R \) and \( Y^G \) in terms of \( Y \) as follows:
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- Define processes \(Y^R\) and \(Y^G\) in terms of \(Y\) as follows:
  - Decompose \(\mathbb{N}\) as a disjoint union, \(\mathbb{N} = \bigcup_{k \geq 1} \{r_k, \ldots s_k\}\),
    \(r_k \leq s_k\),
    \(p_{r_k} = \cdots = p_x = \cdots = p_{s_k}\) and \(p_{s_k} > p_{r_{k+1}}, k \geq 1\).
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\( \circ \) For each \( r_k, s_k \) pair, say \( r, s \) define \( Y^{(r,s)} = (Y_r, \ldots, Y_s) \).
\begin{itemize}
  \item \( \{ p_x : x \in \mathbb{N} \} \), a non-increasing mass function on \( \mathbb{N} \).
  \item \( \hat{p}_{n,x} \equiv n^{-1} \# \{ i \leq n : X_i = x \} \) for \( x \in \mathbb{N} \).
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  \item Define processes \( Y^R \) and \( Y^G \) in terms of \( Y \) as follows:
  \begin{itemize}
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    \item For each \( r_k, s_k \) pair, say \( r, s \) define \( Y^{(r,s)} = (Y_r, \ldots, Y_s) \).
    \item \( Y^R_x = \text{rear}(Y^{(r,s)})_x \) and \( Y^G_x = \text{Gren}(Y^{(r,s)})_x \).
  \end{itemize}
\end{itemize}
Theorem. (Jankowski and Wellner, 2009)

\[(Y_n, Y_n^R, Y_n^G) \Rightarrow (Y, Y^R, Y^G)\]

in \(\ell_2 \times \ell_2 \times \ell_2\) where \(\ell_2 \equiv \{\{y_x\} : \sum_{x\geq 0} y_x^2 < \infty\}\).

Corollary 1. If \(p_{x+1} < p_x\) for all \(x \geq 0\), then

\[(Y_n, Y_n^R, Y_n^G) \Rightarrow (Y, Y, Y)\]

in \(\ell_2 \times \ell_2 \times \ell_2\). In this case the three estimators are asymptotically equivalent.

Corollary 2. If \(p_x = (y + 1)^{-1}1_{\{0,\ldots,y\}}(x)\), then

\[(Y_n, Y_n^R, Y_n^G) \Rightarrow (Y, \text{rear}(Y), \text{Gren}(Y))\),

and ...
\[
E \|Y_n\|_2^2 = nE \left\{ \sum_{x=0}^{y} (\hat{p}_{n,x} - p_x)^2 \right\} \rightarrow E \|Y_x\|_2^2 = 1 - \frac{1}{y + 1},
\]

\[
E \|Y^R_n\|_2^2 = nE \left\{ \sum_{x=0}^{y} (\hat{p}^{\text{rear}}_{n,x} - p_x)^2 \right\} \rightarrow E \|\text{rear}(Y)\|_2^2 = 1 - \frac{1}{y + 1},
\]

\[
E \|Y^G_n\|_2^2 = nE \left\{ \sum_{x=0}^{y} (\hat{p}^{\text{Gren}}_{n,x} - p_x)^2 \right\} \rightarrow E \|\text{Gren}(Y)\|_2^2 = \frac{1}{y + 1}\sum_{x=1}^{y+1} \frac{1}{x} \sim \frac{\log(y + 1)}{y}.
\]

Hence \(\hat{p}^{\text{rear}}_n\) is (asymptotically) inadmissible!
What is the problem?

Proposition. \( \{p_x\} \) is monotone decreasing if and only if it is a mixture of uniform mass functions \((y + 1)^{-1}1_{\{0,\ldots,y\}}(x)\):

\[
p_x = \sum_{y=0}^{\infty} (y + 1)^{-1}1_{\{0,\ldots,y\}}(x)q_y
\]

for some probability mass function \( \{q_y\} \). The inversion formula is given by

\[
q_y = -(y + 1) \Delta p_y \equiv -(y + 1)(p_{y+1} - p_y).
\]

Thus we can define two estimators of \( q \):

\[
\hat{q}_{n,y}^{\text{rear}} \equiv -(y + 1)(\hat{p}_{n,y+1}^{\text{rear}} - \hat{p}_{n,y}^{\text{rear}}),
\]

\[
\hat{q}_{n,y}^{\text{Gren}} \equiv -(y + 1)(\hat{p}_{n,y+1}^{\text{Gren}} - \hat{p}_{n,y}^{\text{Gren}}).
\]
Define processes $Z_n$, $Z^R_n$, $Z^G_n$ by

\[
Z_{n,x} \equiv \sqrt{n}(\hat{q}_{n,x} - q_x), \\
Z^R_{n,x} \equiv \sqrt{n}(\hat{q}_{n,x}^{\text{rearr}} - q_x), \\
Z^G_{n,x} \equiv \sqrt{n}(\hat{q}_{n,x}^{\text{Gren}} - q_x).
\]

We know that if $\sum_{x \geq 0} x^2 p_x = E(X^2) < \infty$, then

\[
Z_n \Rightarrow Z \equiv \{-(x + 1)\Delta Y_x\} \quad \text{in } \ell_2.
\]

- **Problem 1.** If $\sum_{x \geq 0} x^2 p_x < \infty$, does it hold that

\[
Z^R_n \Rightarrow Z^R \equiv \{-(x + 1)\Delta Y^R_x\} \quad \text{in } \ell_2; \\
Z^G_n \Rightarrow Z^G \equiv \{-(x + 1)\Delta Y^G_x\} \quad \text{in } \ell_2?
\]
Problem 2. If \( \{p_x\} \) is strictly decreasing, for what sequences \( a_n, b_n \) (with \( a_n/\sqrt{n} \to \infty \), \( b_n/\sqrt{n} \to \infty \)) does it hold that

\[
\begin{align*}
  a_n \| \hat{p}_n \text{_{rearr}} - \hat{p}_n \|_2 & \to_{p,a.s.} 0, \\
  b_n \| \hat{p}_n \text{_{Gren}} - \hat{p}_n \|_2 & \to_{p,a.s.} 0?
\end{align*}
\]
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• Problem 3. When (or in exactly what senses) does \( \hat{q}_n^{\text{Gren}} \) beat \( \hat{q}_n^{\text{rearr}} \)?
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\[
a_n ||\hat{p}_n^{\text{rearr}} - \hat{p}_n||_2 \to_{p,a.s.} 0, \\
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\]

• Problem 3. When (or in exactly what senses) does \( \hat{q}_n^{\text{Gren}} \) beat \( \hat{q}_n^{\text{rearr}} \)?

• Problem 4. What are the analogues of these results when \( \{p_s\} \) is \( k \)-monotone; i.e. when

\[
p_x = \sum_{y=0}^{\infty} \sum_{x'=0}^{y} \frac{(y - x')^{k-1}}{\sum_{x'=0}^{y} (y - x')^{k-1}} q_y
\]

for some probability mass function \( \{q_y\} \)?
3. Problems 5-6 from Gothenburg meeting

Known from Woodroofe and Sun (1993): in the continuous case, the Grenander estimator $\hat{f}_n$ of a decreasing density is not consistent at zero:

$$\hat{f}_n(0) \to_d f_0(0)Y_1 \equiv f_0(0)\sup_{t>0} \frac{\bar{N}(t)}{t} \overset{d}{=} f_0(0)\mathcal{U}^{-1}$$

where $\mathcal{U} \sim \text{Uniform}(0, 1)$.

Question: If $f_0$ is not bounded at zero, what is the behavior of $\hat{f}_n(0)$?
Theorem. (Balabdaoui, Jankowski, Pavlides, Seregin and W, 2009): Suppose that $F_0$ is regularly varying at 0 with exponent $\gamma \in (0, 1]$. Then with $a_n$ satisfying $nF_0(a_n) \to 1$ as $n \to \infty$,

$$na_n f_n(ta_n) \Rightarrow \hat{h}_\gamma(t) \quad \text{in } D[0, \infty)$$

where $\hat{h}_\gamma$ is the right derivative of the least concave majorant of $N(t^\gamma)$ and $N$ is a standard Poisson process.

Now suppose that $f_0$ is $k-$monotone on $(0, \infty)$ with $k \geq 2$; i.e.

$$f(x) = \int_0^\infty \frac{1}{y^k} (y - x)^{k-1} dG(y)$$

for some probability distribution $G$.

Problem 5. If $f_0$ is $k-$monotone, what is the behavior of $\hat{f}_n(0)$?

Problem 6. If $f_0$ is completely monotone (i.e. representable as a scale mixture of exponentials), what is the behavior of $\hat{f}_n(0)$?
4. Four more problems involving shape constraints ... very briefly