Find the $n^{th}$ percentile of an exponential distribution with parameter $\lambda$. Hint: Answer will depend on $\lambda$ and $n$.

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

The $n^{th}$ percentile is given by:

$$\int_0^{n^{th} \text{ percentile}} \lambda e^{-\lambda x} \, dx = \frac{n}{100}$$

Solving for the $n^{th}$ percentile:

$$\frac{\lambda e^{-\lambda x}}{-\lambda} \bigg|_0^{n^{th} \text{ percentile}} = \frac{n}{100} \Rightarrow \lambda \left( e^{-\lambda (n^{th} \text{ percentile})} - e^{-\lambda (0)} \right) = \frac{n}{100} \Rightarrow e^{-\lambda (n^{th} \text{ percentile})} = 1 - \frac{n}{100}$$

$$-\lambda (n^{th} \text{ percentile}) = \log \left( 1 - \frac{n}{100} \right) \Rightarrow n^{th} \text{ percentile} = -\frac{1}{\lambda} \log \left( 1 - \frac{n}{100} \right)$$

The Poisson mass function in the above example is "flat" at the top, i.e., $p(x)$ has the same value at $x=3$ and $x=4$. Show that, quite generally, the Poisson mass function has the same value at $x=2$ (i.e., at the average) and at $x=(2-1)$.

$$p(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$p(x=3) = \frac{e^{-\lambda} \lambda^3}{3!} = \frac{e^{-\lambda} \lambda^3}{2 (2-1)!} = \frac{e^{-\lambda} \lambda^3}{(2-1)!}$$

$$p(x=2-1) = \frac{e^{-\lambda} \lambda^{2-1}}{(2-1)!}$$
Chapter 1

Consider the examples of Poisson in lecture.

a) Find another example (google, books,...) that qualifies as a Poisson variable. Call it $X$, and define it clearly.

b) Assume, or even guess, what the value of the $\lambda$ parameter may be for your example. Remember, $\lambda$ is the average $X$. State that value, with the correct units.

c) Plot the Poisson dist. with that value of $\lambda$ (by R or by hand).

d) Compute $P(x=0)$, and interpret it.

a) Students' answers will vary. But any variable that is a count per unit length, time, area, etc. is likely to be Poisson.

b) $\lambda$ can be approximated (estimated) with the sample mean of $X$. You can guess it! For the lecture examples:

1) # of knots per feet of wood. $\sim 2$ per feet
2) # of car crashes at some intersection, per month $\sim 5$ per month

$c)$ $x=0,1,2,\ldots$ $P(x)=e^{-\lambda}\frac{\lambda^x}{x!}$

$d)$ $P(x=0)=e^{-\lambda}\frac{\lambda^0}{0!}=e^{-\lambda}$

1) prop. of feet of wood which will have no knots.
2) "" months that will have no accidents.
Consider the binomial distribution with parameters $n=4, \pi = 1/4$.

a) Compute specific values of $p(x)$ for all possible values of $x$. (By hand or By R).

b) Compute $E[x] = \sum_x x \cdot p(x)$, and compare the answer with the value of $(n\pi)$. (By hand or By R).

c) Take a sample of size 100 from $p(x)$, compute the sample mean of the 100 numbers, and compare the answer with the answer in part b.

\[
p(x) = \binom{4}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{4-x}
\]

\[
p(x=0) = \binom{4}{0} \left(\frac{3}{4}\right)^4 = \frac{4!}{0!4!} \left(\frac{3}{4}\right)^4 = \left(\frac{3}{4}\right)^4 = 0.3164
\]

\[
p(x=1) = \binom{4}{1} \frac{3}{4} \left(\frac{3}{4}\right)^3 = \frac{4!}{1!3!} \left(\frac{3}{4}\right)^1 = \frac{4}{3} \left(\frac{3}{4}\right)^4 = 0.4218
\]

\[
p(x=2) = \binom{4}{2} \frac{3^2}{4^2} = \frac{4!}{2!2!} \left(\frac{3}{4}\right)^2 \frac{1}{2} = 6 \left(\frac{3}{4}\right)^4 \frac{1}{2} = \frac{2}{3} \left(\frac{3}{4}\right)^4 = 0.2109
\]

\[
p(x=3) = \binom{4}{3} \frac{3^3}{4^3} = 4 \cdot \frac{3}{4} \left(\frac{3}{4}\right)^3 = \frac{4}{27} \left(\frac{3}{4}\right)^4 = 0.0469
\]

\[
p(x=4) = \binom{4}{4} \frac{3^4}{4^4} = \frac{1}{4^4} = 0.0009
\]

\[
\sum_{x=0}^4 x \cdot p(x) = 0(0.3164) + 1(0.4218) + 2(0.2109) + 3(0.0469) + 4(0.0009) = 0.9999 \quad \text{< Same.}
\]

\[
E[x] = n \pi = 4 \cdot \frac{1}{4} = 1
\]

\[
x = \text{rbinom}(100, 4, 1/4) \Rightarrow x = 3, 1, 2, 1, 1, 1, 2, \ldots
\]

\[
\text{mean}(x) \Rightarrow 0.91 \quad \text{(These answers will vary across students/)}
\]

\[
C \quad \text{approximately equal to 1. Make sense?}
\]
For the uniform distribution between $a, b$, show that the expected value is $\frac{1}{2} (a+b)$.

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{else} \end{cases}$$

$$E(x) = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{a}^{b} x \cdot \frac{1}{b-a} \, dx$$

$$= \frac{1}{b-a} \int_{a}^{b} x \, dx = \frac{1}{b-a} \left[ \frac{1}{2} x^2 \right]_{a}^{b}$$

$$= \frac{1}{b-a} \left[ \frac{1}{2} (b^2-a^2) \right] = \frac{1}{2} \cdot \frac{1}{b-a} (b-a)(b+a) = \frac{1}{2} (b+a)$$

Find the Exp. Value of the exponential distribution with parameter $\lambda$.

Hint: $\int_{0}^{\infty} ye^{-y} \, dy = 1$.

For exponential distribution: $f(x) = 1 \cdot e^{-\lambda x}$. Then,

$$E(x) = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{0}^{\infty} xe^{-\lambda x} \, dx$$

$$= \frac{1}{\lambda} \int_{0}^{\infty} e^{-y} \, dy = \frac{1}{\lambda}.$$
Find the $\mu_x$ for

a) The $p(x)$ given in exercise 1.27, with the two "?" given as 0.1, and zero, respectively.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>0.1</td>
<td>0.15</td>
<td>0.2</td>
<td>0.25</td>
<td>0.2</td>
<td>0.1</td>
<td>0</td>
</tr>
</tbody>
</table>

$\mu_x = \sum x \cdot p(x) = (0 \cdot 0.1 + 1 \cdot 0.15 + 2 \cdot 0.2 + 3 \cdot 0.25 + 4 \cdot 0.2 + 5 \cdot 0.1 + 6 \cdot 0) = 2.6$

b) The $f(x)$ given in exercise 1.22. $f(x) = \begin{cases} 0.04 x & 0 < x \leq 5 \\ 0.4 - 0.04 x & 5 < x \leq 10 \\ 0 & \text{else} \end{cases}$

$\mu_x = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{0}^{5} x \cdot 0.04 \, dx + \int_{5}^{10} x \cdot (0.4 - 0.04x) \, dx + 0$

$= 0.04 \int_{0}^{5} x^2 \, dx + 0.4 \int_{5}^{10} x \, dx - 0.04 \int_{5}^{10} x^2 \, dx$

$= 0.04 \frac{1}{3} x^3 \int_{0}^{5} + 0.4 \left[ \frac{x^2}{2} \right]_{5}^{10} - 0.04 \frac{1}{3} x^3 \bigg|_{5}^{10}$

$= 0.04 \frac{1}{3} (125 - 0) + 0.4 \frac{1}{2} (100 - 25) - 0.04 \frac{1}{3} (1000 - 125)$

$= \frac{5}{3} + 15 - \frac{35}{3} = 15 - \frac{30}{3} = \boxed{5}$ which makes sense.

$\text{f(x)}$