Design of Engineering Experiments Part 10 – Nested and Split-Plot Designs

- Text reference, Chapter 14, Pg. 525
- These are **multifactor** experiments that have some important industrial applications
- Nested and split-plot designs frequently involve one or more **random** factors, so the methodology of Chapter 13 (expected mean squares, variance components) is important
- There are **many** variations of these designs – we consider only some basic situations

Two-Stage Nested Design

- Section 14-1 (pg. 525)
- In a nested design, the levels of one factor (*B*) is **similar** to but not **identical** to each other at different levels of another factor (*A*)
- Consider a company that purchases material from three suppliers
  - The material comes in batches
  - Is the purity of the material uniform?
- Experimental design
  - Select four batches at random from each supplier
  - Make three purity determinations from each batch

Two-Stage Nested Design

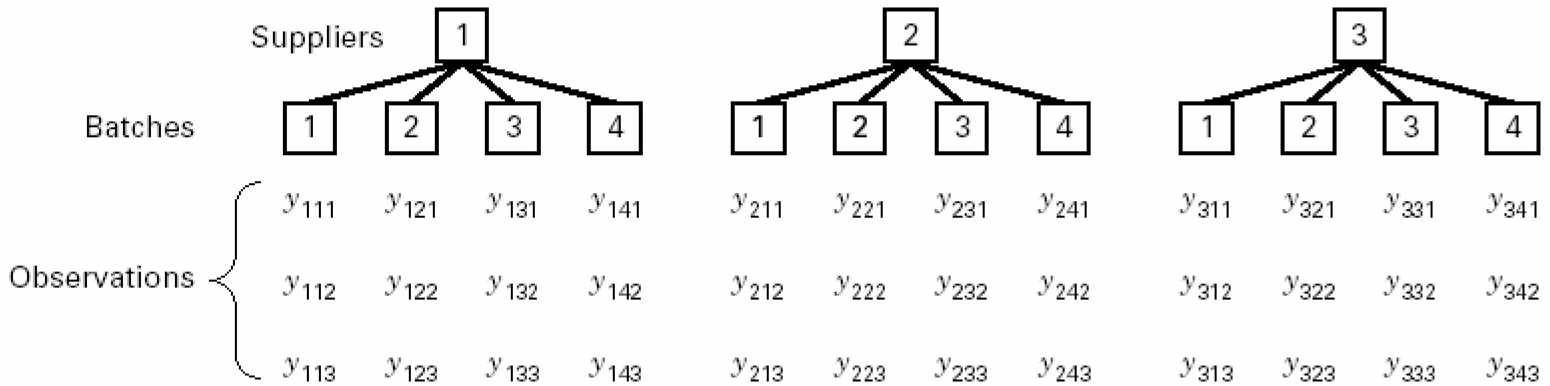

Figure 14-1 A two-stage nested design.

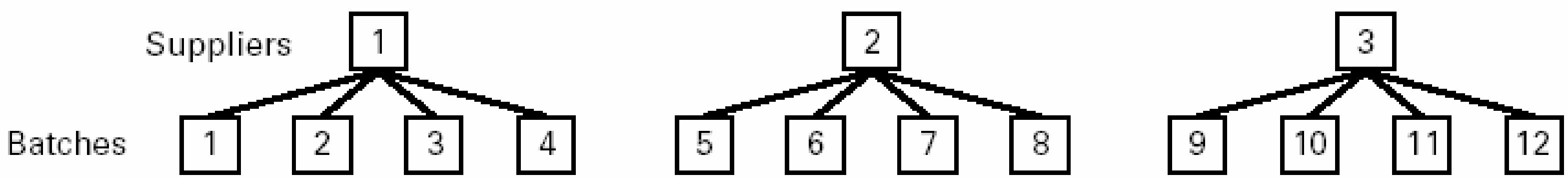

Figure 14-2 Alternate layout for the two-stage nested design.

Two-Stage Nested Design Statistical Model and ANOVA

$$y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \varepsilon_{(ij)k} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

$$SS_T = SS_A + SS_{B(A)} + SS_E$$

$$df : abn - 1 = a - 1 + a(b-1) + ab(n-1)$$

Table 14-1 Expected Mean Squares in the Two-Stage Nested Design

$E(MS)$	A Fixed B Fixed	A Fixed B Random	A Random B Random
$E(MS_A)$	$\sigma^2 + \frac{bn \sum \tau_i^2}{a-1}$	$\sigma^2 + n\sigma_\beta^2 + \frac{bn \sum \tau_i^2}{a-1}$	$\sigma^2 + n\sigma_\beta^2 + bn\sigma_\tau^2$
$E(MS_{B(A)})$	$\sigma^2 + \frac{n \sum \sum \beta_{j(i)}^2}{a(b-1)}$	$\sigma^2 + n\sigma_\beta^2$	$\sigma^2 + n\sigma_\beta^2$
$E(MS_E)$	σ^2	σ^2	σ^2

Two-Stage Nested Design Example 14-1 (pg. 528)

Three suppliers, four batches (selected randomly) from each supplier, three samples of material taken (at random) from each batch

Experiment and data, Table 14-3

Data is coded

Minitab balanced ANOVA will analyze nested designs

Mixed model, assume restricted form

Minitab Analysis – Page 530

Table 14-6 Minitab Output (Balanced ANOVA) for Example 14-1

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Analysis of Variance (Balanced Designs)

Factor              Type   Levels  Values
Supplier           fixed        3       1      2      3
Batch(Supplier)   random        4       1      2      3      4

Analysis of Variance for Purity

Source              DF        SS       MS      F       P
Supplier             2    15.056    7.528   0.97   0.416
Batch(Supplier)      9    69.917    7.769   2.94   0.017
Error               24    63.333    2.639
Total               35   148.306

Source               Variance Error Expected Mean Square for Each Term
                    component term  (using restricted model)
1 Supplier                       2  (3) + 3(2) + 12Q[1]
2 Batch(Supplier)       1.710    3  (3) + 3(2)
3 Error                 2.639       (3)
```

Practical Interpretation – Example 14-1

- There is no difference in purity among suppliers, but significant difference in purity among batches (within suppliers)
- What are the practical implications of this conclusion?
- Examine residual plots – pg. 532 – plot of residuals versus supplier is very important (why?)
- What if we had incorrectly analyzed this experiment as a factorial? (see Table 14-5, pg. 529)
- Estimation of variance components (pg. 532)

Variations of the Nested Design

- Staggered nested designs (Pg. 533)
  - Prevents too many degrees of freedom from building up at lower levels
  - Can be analyzed in Minitab (General Linear Model) – see the supplemental text material for an example
- Several levels of nesting (pg. 534)
  - The alloy formulation example
  - This experiment has three stages of nesting
- Experiments with both nested and “crossed” or factorial factors (pg. 536)

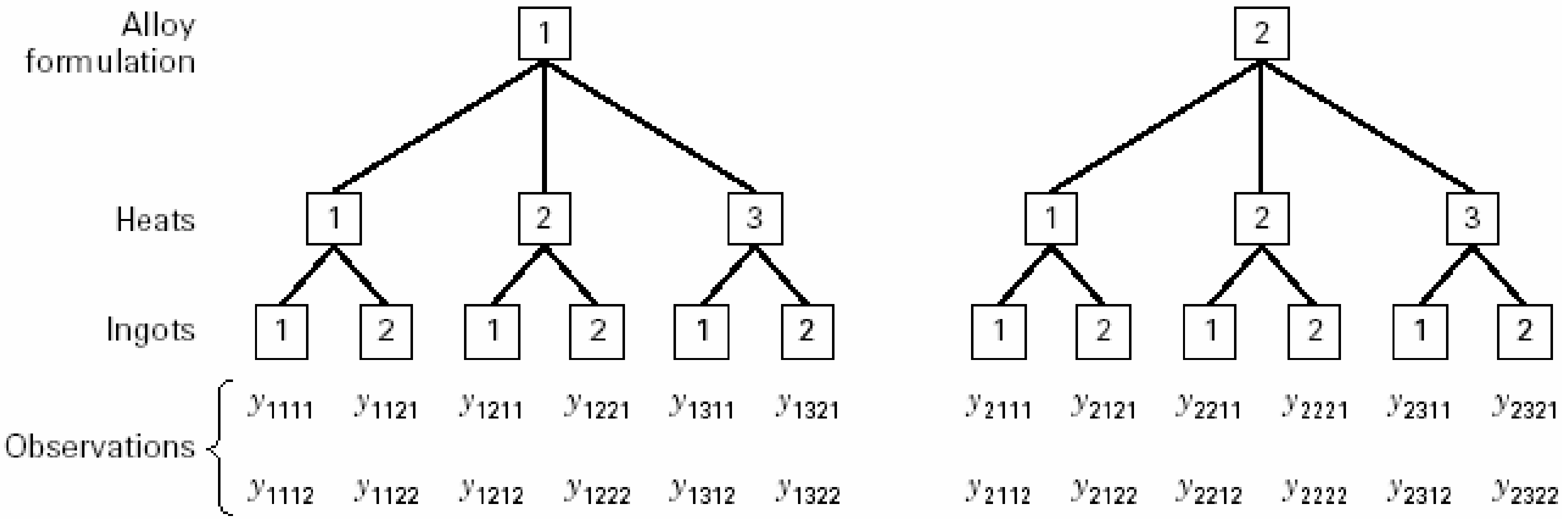


Figure 14-5 A three-stage nested design.

Example 14-2 Nested and Factorial Factors

Table 14-9 Assembly Time Data for Example 14-2

	Layout 1				Layout 2				
Operator	1	2	3	4	1	2	3	4	$y_{i...}$
Fixture 1	22	23	28	25	26	27	28	24	404
	24	24	29	23	28	25	25	23	
Fixture 2	30	29	30	27	29	30	24	28	447
	27	28	32	25	28	27	23	30	
Fixture 3	25	24	27	26	27	26	24	28	401
	21	22	25	23	25	24	27	27	
Operator totals, $y_{.jk.}$	149	150	171	149	163	159	151	160	
Layout totals, $y_{.j..}$	619				633				$1252 = y_{....}$

$$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_{k(j)} + (\tau\beta)_{ij} + (\tau\gamma)_{ik(j)} + \varepsilon_{(ijk)l} \quad \begin{cases} i = 1,2,3 \\ j = 1,2 \\ j = 1,2,3,4 \\ l = 1,2 \end{cases}$$

Example 14-2 – Expected Mean Squares

Assume that fixtures and layouts are fixed, operators are random – gives a **mixed** model (use restricted form)

Table 14-10 Expected Mean Squares for Example 14-2

Model Term	Expected Mean Square
τ_i	$\sigma^2 + 2\sigma^2_{\tau\gamma} + 8\sum \tau_i^2$
β_j	$\sigma^2 + 6\sigma^2_{\gamma} + 24\sum \beta_j^2$
$\gamma_{k(j)}$	$\sigma^2 + 6\sigma^2_{\gamma}$
$(\tau\beta)_{ij}$	$\sigma^2 + 2\sigma^2_{\tau\gamma} + 4\sum\sum (\tau\beta)^2_{ij}$
$(\tau\gamma)_{ik(j)}$	$\sigma^2 + 2\sigma^2_{\tau\gamma}$
$\epsilon_{(ijk)l}$	σ^2

Example 13-2 – Minitab Analysis

Table 14-12 Minitab Balanced ANOVA Analysis of Example 14-2 Using the Restricted Model

Analysis of Variance (Balanced Designs)

Factor	Type	Levels	Values			
Layout	fixed	2	1	2		
Operator(Layout)	random	4	1	2	3	4
Fixture	fixed	3	1	2	3	

Analysis of Variance for Time

Source	DF	SS	MS	F	P
Layout	1	4.083	4.083	0.34	0.581
Operator(Layout)	6	71.917	11.986	5.14	0.002
Fixture	2	82.792	41.396	7.55	0.008
Layout*Fixture	2	19.042	9.521	1.74	0.218
Fixture*Operator(Layout)	12	65.833	5.486	2.35	0.036
Error	24	56.000	2.333		
Total	47	299.667			

Source	Variance component	Error term	Expected Mean Square for Each Term (using restricted model)
1 Layout		2	(6) + 6(2) + 24Q[1]
2 Operator(Layout)	1.609	6	(6) + 6(2)
3 Fixture		5	(6) + 2(5) + 16Q[3]
4 Layout*Fixture		5	(6) + 2(5) + 8Q[4]
5 Fixture*Operator(Layout)	1.576	6	(6) + 2(5)
6 Error	2.333		(6)

The Split-Plot Design

- Text reference, Section 14-4 page 540
- The split-plot is a multifactor experiment where it is not possible to completely randomize the order of the runs
- Example – paper manufacturing
  - Three pulp preparation methods
  - Four different temperatures
  - Each replicate requires 12 runs
  - The experimenters want to use three replicates
  - How many batches of pulp are required?

The Split-Plot Design

- Pulp preparation methods is a **hard-to-change** factor
- Consider an alternate experimental design:
  - In replicate 1, select a pulp preparation method, prepare a batch
  - Divide the batch into four sections or samples, and assign one of the temperature levels to each
  - Repeat for each pulp preparation method
  - Conduct replicates 2 and 3 similarly

The Split-Plot Design

- Each replicate (sometimes called **blocks**) has been divided into three parts, called the **whole plots**
- Pulp preparation methods is the **whole plot treatment**
- Each whole plot has been divided into four **subplots** or **split-plots**
- Temperature is the **subplot treatment**
- Generally, the hard-to-change factor is assigned to the whole plots
- This design requires only 9 batches of pulp (assuming three replicates)

The Split-Plot Design
Model and Statistical Analysis

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_k + (\tau\gamma)_{ik} + (\beta\gamma)_{jk}$$

$$+(\tau\beta\gamma)_{ijk} + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, r \\ j = 1, 2, \ldots, a \\ k = 1, 2, \ldots, b \end{cases}$$

There are two error structures; the whole-plot error and the subplot error

Table 14-15 Expected Mean Squares for Split-Plot Design

	Model Term	Expected Mean Square
Whole plot	τ_i	$\sigma^2 + ab\sigma_\tau^2$
	β_j	$\sigma^2 + b\sigma_{\tau\beta}^2 + \frac{rb \sum \beta_j^2}{a-1}$
	$(\tau\beta)_{ij}$	$\sigma^2 + b\sigma_{\tau\beta}^2$
Subplot	γ_k	$\sigma^2 + a\sigma_{\tau\gamma}^2 + \frac{ra \sum \gamma_k^2}{(b-1)}$
	$(\tau\gamma)_{ik}$	$\sigma^2 + a\sigma_{\tau\gamma}^2$
	$(\beta\gamma)_{jk}$	$\sigma^2 + \sigma_{\tau\beta\gamma}^2 + \frac{r \sum \sum (\beta\gamma)_{jk}^2}{(a-1)(b-1)}$
	$(\tau\beta\gamma)_{ijk}$	$\sigma^2 + \sigma_{\tau\beta\gamma}^2$
	$\epsilon_{(ijk)h}$	σ^2 (not estimable)

Split-Plot ANOVA

Table 14-16 Analysis of Variance for the Split-Plot Design Using the Tensile Strength Data from Table 14-14

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Replicates (or Blocks)	77.55	2	38.78		
Preparation method (A)	128.39	2	64.20	7.08	0.05
Whole Plot Error (replicates (or Blocks) $\times A$)	36.28	4	9.07		
Temperature (B)	434.08	3	144.69	41.94	<0.01
Replicates (or Blocks) $\times B$	20.67	6	3.45		
AB	75.17	6	12.53	2.96	0.05
Subplot Error (replicates (or Blocks) $\times AB$)	50.83	12	4.24		
Total	822.97	35			

Calculations follow a three-factor ANOVA with one replicate

Note the two different **error structures**; whole plot and subplot

Alternate Model for the Split-Plot

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \gamma_k + (\beta\gamma)_{jk} + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, r \\ j = 1, 2, \ldots, a \\ k = 1, 2, \ldots, b \end{cases}$$

Factor	$E(MS)$
τ_i (Replicates or Blocks)	$\sigma_\epsilon^2 + ab\sigma_\tau^2$
β_j (A)	$\sigma_\epsilon^2 + b\sigma_{\tau\beta}^2 + \dfrac{rb\sum \beta_j^2}{a-1}$
$(\tau\beta)_{ij}$	$\sigma_\epsilon^2 + b\sigma_{\tau\beta}^2$ (whole plot error)
γ_k (B)	$\sigma_\epsilon^2 + \dfrac{ra\sum \gamma_k^2}{ab-1}$
$(\beta\gamma)_{jk}$ (AB)	$\sigma_\epsilon^2 + \dfrac{r\sum\sum (\beta\gamma)_{jk}^2}{(a-1)(b-1)}$
ϵ_{ijk}	σ_ϵ^2 (subplot error)

Variations of the basic split-plot design

More than two factors – see page 545

A & *B* (gas flow & temperature) are hard to change;
C & *D* (time and wafer position) are easy to change.

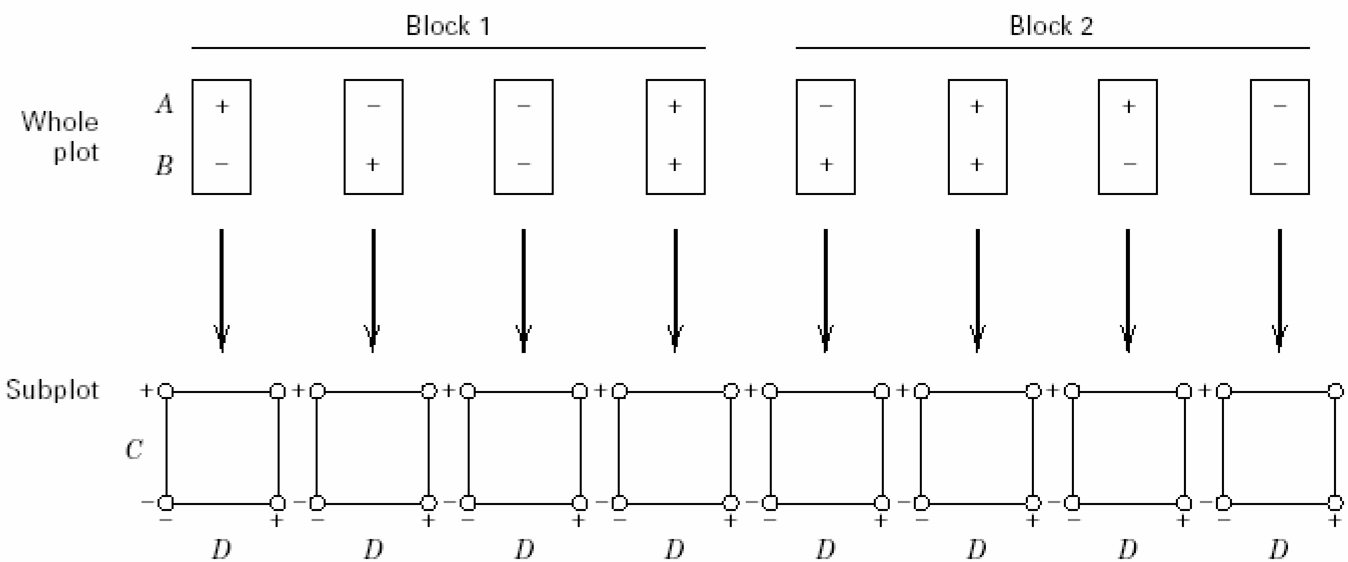


Figure 14-7 A split-plot design with four design factors, two in the whole plot and two in the subplot.

Unreplicated designs and fractional factorial design in a split-plot framework

EXAMPLE 14-3

The factors affecting uniformity in a single-wafer plasma etching process are being investigated. Three factors on the etching tool are relatively difficult to change from run-to-run: A = electrode gap, B = gas flow, and C = pressure. Two other factors are easy to change from run to run: $\boldsymbol{D}$ = time and $\boldsymbol{E}$ = RF (radio frequency) power. The experimenters want to use a fractional factorial experiment to investigate these five factors because the number of test wafers available is limited. The hard-to-change factors also indicate that a split-plot design should be considered. The experimenters decide to use the strategy discussed above: a 2^{5-1} design with factors A, B, and C in the whole plots and factors $\boldsymbol{D}$ and $\boldsymbol{E}$ in the subplots. The design generator is $\boldsymbol{E} = ABC\boldsymbol{D}$. This produces a 16-run fractional factorial with eight whole plots. Every whole plot contains one of the eight treatment combinations from a complete 2^3 factorial design in factors A, B and C. Each whole plot is divided into two subplots, with one of the treatment combinations for factors $\boldsymbol{D}$ and $\boldsymbol{E}$ in each subplot. The design and the resulting uniformity data are shown in Table 14-19. The eight whole plots were run in random order, but once a whole plot configuration for factors A, B, and C was set up on the etching tool, both subplot runs were made (also in random order).

Table 14-19 The 2^{5-1} Split-Plot Experiment for the Plasma Etching Tool

Whole Plots	Whole-Plot Factors *A*	*B*	*C*	Subplot Factors *D*	*E*	Uniformity
	−	−	−	−	+	40.85
1	−	−	−	+	−	41.07
	+	−	−	−	−	35.67
2	+	−	−	+	+	51.15
	−	+	−	−	−	41.80
3	−	+	−	+	+	37.01
	+	+	−	−	+	91.09
4	+	+	−	+	−	48.67
	−	−	+	−	−	40.32
5	−	−	+	+	+	43.34
	+	−	+	−	+	62.46
6	+	−	+	+	−	38.08
	−	+	+	−	+	31.99
7	−	+	+	+	−	41.03
	+	+	+	−	−	70.31
8	+	+	+	+	+	81.03

The statistical analysis of this experiment involves keeping the whole plot and subplot factors separate. We assume that all interactions beyond order two are negligible. Figure 14-8*a* is a half-normal probability plot of the effect estimates for only the whole-plot factors, ignoring the factors in the subplots. Notice that factors *A*, *B*, and the *AB* interaction have large effects. Figure 14-8*b* is a half-normal probability plot of the subplot effects ***D*** and ***E***, and the interactions involving those factors, ***DE***, ***AD***, ***AE***, ***BD***, ***BE***, ***CD***, and ***CE***. Only the main effect of ***E*** and the ***AE*** interaction are large.

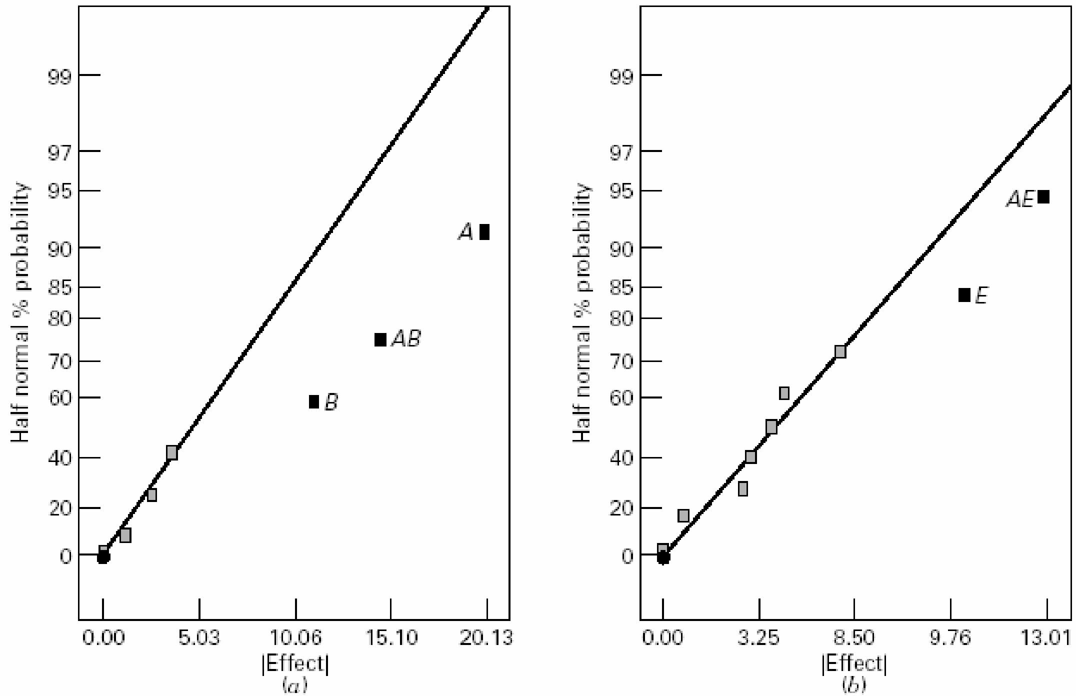


Figure 14-8 Half-normal plots of the effects from the 2^{5-1} split-plot experiment in Example 14-3. (*a*) Whole plot effects. (*b*) Subplot effects.

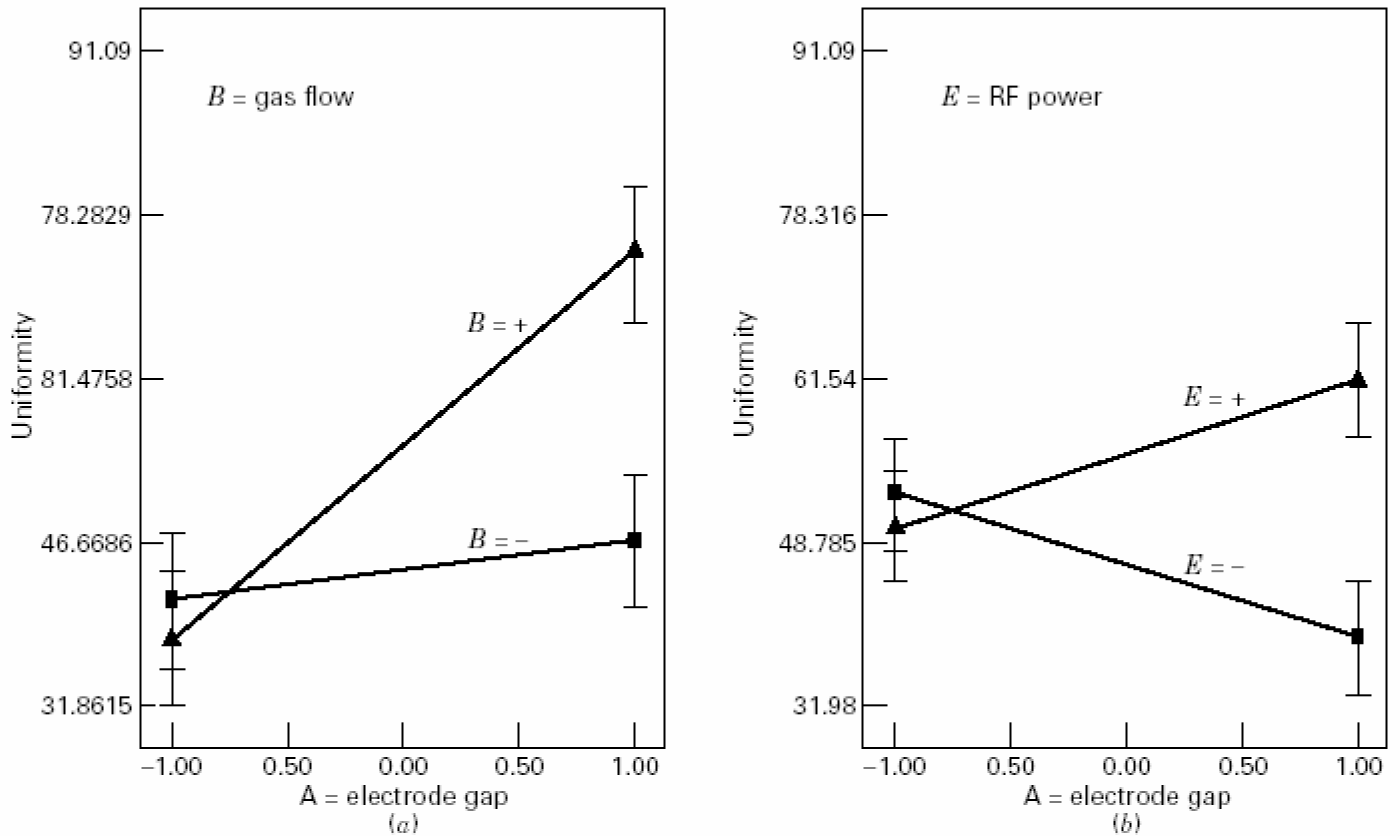
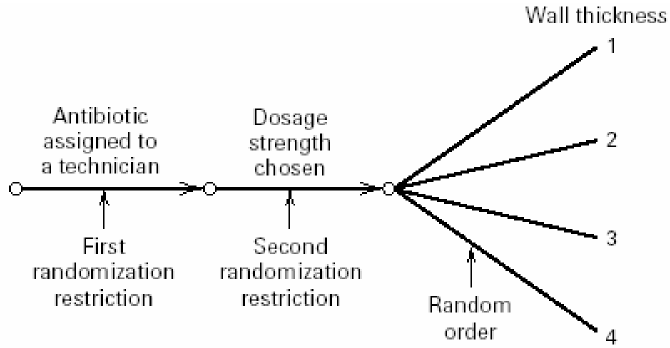


Figure 14-9 Two-factor interaction graphs for the 2^{5-1} split-plot experiment in Example 14-3. (*a*) *AB* interaction. (*b*) *AE* interaction.



		Technician								
		1			2			3		
Blocks	Dosage strength	1	2	3	1	2	3	1	2	3
1	Wall thicknesses	1	1	1	1	1	1	1	1	1
		2	2	2	2	2	2	2	2	2
		3	3	3	3	3	3	3	3	3
		4	4	4	4	4	4	4	4	4
2	Wall thicknesses	1	1	1	1	1	1	1	1	1
		2	2	2	2	2	2	2	2	2
		3	3	3	3	3	3	3	3	3
		4	4	4	4	4	4	4	4	4
3	Wall thicknesses	1	1	1	1	1	1	1	1	1
		2	2	2	2	2	2	2	2	2
		3	3	3	3	3	3	3	3	3
		4	4	4	4	4	4	4	4	4
4	Wall thicknesses	1	1	1	1	1	1	1	1	1
		2	2	2	2	2	2	2	2	2
		3	3	3	3	3	3	3	3	3
		4	4	4	4	4	4	4	4	4

Figure 14-10 A split-split-plot design.

A split-split-plot design

Two randomization restrictions present within each replicate

The strip-split-plot design

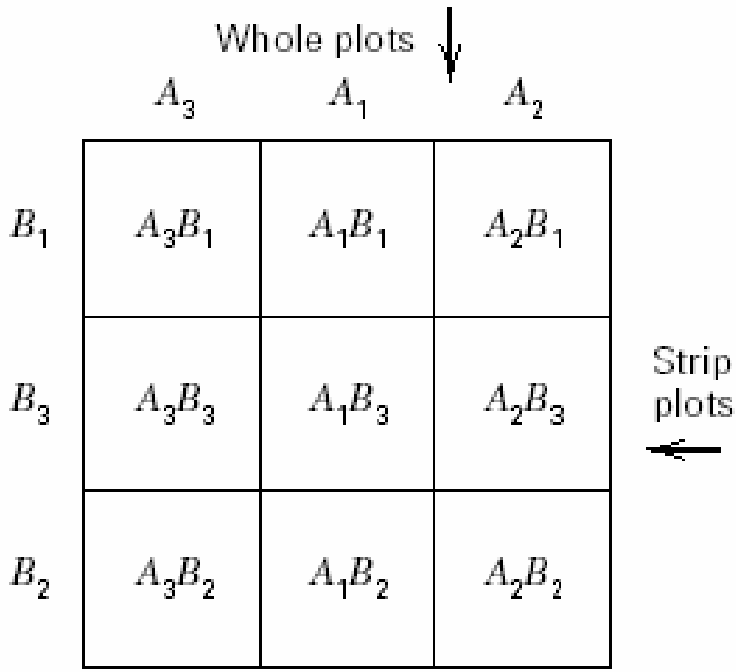


Figure 14-11 One replicate (block) of a strip-split-plot design.

The "strips" are just another set of whole plots