

Applications of the Noncentral t-Distribution

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1. Introduction.

This report provides background information and some limited guidance in using the FORTRAN subroutines HSPNCT and HSPINT in several typical applications. These routines evaluate, respectively, the noncentral t-distribution function and its inverse.

The noncentral t-distribution is intimately tied to statistical inference procedures for samples from normal populations. For simple random samples from a normal population the usage of the noncentral t-distribution includes basic power calculations, variables acceptance sampling plans (MIL-STD-414) and confidence bounds for percentiles, tail probabilities, statistical process control parameters C_L , C_U and C_{pk} and for coefficients of variation. The purpose of this report is to describe these applications in some detail, giving sufficient theoretical derivation so that these procedures may easily be extended to more complex normal data structures, that occur, for example, in multiple regression and analysis of variance settings. We begin by giving a working definition of the noncentral t-distribution, i.e., a definition that ties directly into all the applications. This is demonstrated upfront by exhibiting the basic probabilistic relationship underlying all these applications. Separate sections deal with each of the applications outlined above. The individual sections contain no references. However, a short list is provided to give an entry into the literature on the noncentral t-distribution.

Detailed usage information for HSPNCT and HSPINT is given in attachment A of this report. For current availability information contact the Math/Stat Libraries Project Manager, M/S 7L-21.

The user can use these two subprograms without necessarily reading the detailed explanations of the mathematical basis contained in this report.

2. Definition of the Noncentral t-Distribution

If Z and V are (statistically) independent standard normal and chi-square random variables respectively, the latter with f degrees of freedom, then the ratio

$$T_{f, \delta} = \frac{Z + \delta}{\sqrt{V/f}}$$

is said to have a noncentral t-distribution with f degrees of freedom and noncentrality parameter δ . Here $f \geq 1$ is an integer and δ may be any real number. The cumulative distribution function of $T_{f, \delta}$ is denoted by $G_{f, \delta}(t) = P(T_{f, \delta} \leq t)$. If $\delta = 0$, then the noncentral t-distribution reduces to the usual central or Student t-distribution. $G_{f, \delta}(t)$ increases from 0 to 1 as t increases from $-\infty$ to $+\infty$ or as δ decreases from $+\infty$ to $-\infty$. There appears to be no such simple monotonicity relationship with regard to the parameter f .

Since most of the applications to be treated here concern single samples from a normal population, we will review some of the relevant normal sampling theory. Suppose X_1, \dots, X_n is a random sample from a normal population with mean μ and standard deviation σ . The sample mean \bar{X} and sample standard deviation S are respectively defined as:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad \text{and} \quad S = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2}.$$

The following distributional facts are well known:

- \bar{X} and S are statistically independent;
- \bar{X} is distributed like a normal random variable with mean μ and standard deviation σ/\sqrt{n} , or equivalently, $Z = \sqrt{n}(\bar{X} - \mu)/\sigma$ has a standard normal distribution (mean = 0 and standard deviation = 1);
- $V = (n-1)S^2/\sigma^2$ has a chi-square distribution with $f = n-1$ degrees of freedom and is statistically independent of Z .

All one-sample applications involving the noncentral t-distribution can be reduced to calculating the following probability

$$\gamma = P(\bar{X} - aS \leq b).$$

To relate this probability to the noncentral t-distribution note the equivalence of the following three inequalities, which can be established by simple algebraic manipulations:

$$\bar{X} - aS \leq b$$

$$\frac{\sqrt{n}(\bar{X} - \mu)/\sigma - \sqrt{n}(b - \mu)/\sigma}{S/\sigma} \leq a\sqrt{n}$$

$$T_{f, \delta} \stackrel{\text{def}}{=} \frac{Z + \delta}{\sqrt{V/f}} \leq a\sqrt{n}$$

with $f = n - 1$, $\delta = -\sqrt{n}(b - \mu)/\sigma$, and with Z and V defined above in terms of \bar{X} and S . Thus

$$\gamma = P(T_{f, \delta} \leq a\sqrt{n}) = G_{f, \delta}(a\sqrt{n}).$$

Depending on the application, three of the four parameters n , a , δ and γ are usually given and the fourth needs to be determined either by direct computation of $G_{f, \delta}(t)$ or by root solving techniques.

3. Power of the t-Test

Assuming the normal sampling situation described above, the following testing problem is often encountered. A hypothesis $H : \mu \leq \mu_0$ is tested against the alternative $A : \mu > \mu_0$. Here μ_0 is some specified value. For testing H against A on the basis of the given sample, the intuitive and in many ways optimal procedure is to reject H in favor of A whenever

$$\frac{\sqrt{n}(\bar{X} - \mu_0)}{S} \geq t_{n-1}(1 - \alpha)$$

or equivalently when

$$\bar{X} - \frac{t_{n-1}(1 - \alpha) S}{\sqrt{n}} \geq \mu_0.$$

Here $t_{n-1}(1-\alpha)$ is the $1-\alpha$ percentile of the central t -distribution with $n-1$ degrees of freedom. In this form the test has chance α or less of rejecting H when $\mu \leq \mu_0$, i.e., when H is true. As will become clear below, the chance of rejection is $< \alpha$ when $\mu < \mu_0$. Thus α is the maximum chance of rejecting H falsely, i.e., the maximum type I error probability.

An important characteristic of a test is its power function, which is defined as the probability of rejecting H as a function of (μ, σ) , i.e.,

$$\beta(\mu, \sigma) = P_{\mu, \sigma} \left(\frac{\sqrt{n}(\bar{X} - \mu_0)}{S} \geq t_{n-1}(1 - \alpha) \right) .$$

The arguments and subscripts (μ, σ) indicate that the probability is calculated assuming that the sample X_1, \dots, X_n comes from a normal population with mean μ and standard deviation σ . For $\mu > \mu_0$ the value of $1 - \beta(\mu, \sigma)$ represents the probability of falsely accepting H , i.e., the probability of type II error. The power function can be expressed directly in terms of $G_{f, \delta}(t)$ by noting

$$\frac{\sqrt{n}(\bar{X} - \mu_0)}{S} = \frac{\sqrt{n}(\bar{X} - \mu)/\sigma + \sqrt{n}(\mu - \mu_0)/\sigma}{S/\sigma} = \frac{Z + \delta}{\sqrt{V/(n-1)}} ,$$

so that

$$\beta(\mu, \sigma) = P_{\mu, \sigma} \left(\frac{\sqrt{n}(\bar{X} - \mu_0)}{S} \geq t_{n-1}(1 - \alpha) \right) = 1 - G_{n-1, \delta}(t_{n-1}(1 - \alpha)) ,$$

where $\delta = \sqrt{n}(\mu - \mu_0)/\sigma$.

In a similar fashion one can deal with the dual problem of testing the hypothesis $H' : \mu \geq \mu_0$ against the alternative $A' : \mu < \mu_0$. The modifications, which consist of reversing certain inequalities, are straightforward and omitted.

For the two-sided problem of testing $H^* : \mu = \mu_0$ against the alternative $A^* : \mu \neq \mu_0$ the relevant test rejects H^* in favor of A^* whenever

$$\frac{\sqrt{n}|\bar{X} - \mu_0|}{S} \geq t_{n-1}(1 - \alpha/2) .$$

The power function of this test is calculated along the same lines as before as

$$P_{\mu, \sigma} \left(\frac{\sqrt{n}(\bar{X} - \mu_0)}{S} \leq -t_{n-1}(1 - \alpha/2) \text{ or } \frac{\sqrt{n}(\bar{X} - \mu_0)}{S} \geq t_{n-1}(1 - \alpha/2) \right)$$

$$= 1 - G_{n-1, \delta}(t_{n-1}(1 - \alpha/2)) + G_{n-1, \delta}(-t_{n-1}(1 - \alpha/2)) = \beta^*(\mu, \sigma),$$

where $\delta = \sqrt{n}(\mu - \mu_0)/\sigma$ as before.

4. Variables Acceptance Sampling Plans

Quality control applications governed by MIL-STD-414 deal with variables acceptance sampling plans (VASP). In a VASP the quality of items in a given sample is measured on a quantitative scale. An item is judged defective when its measured quality exceeds a certain threshold.

The samples are drawn randomly from a population of items. The objective is to make inferences about the proportion of defectives in the population. This leads either to an acceptance or a rejection of the population quality as a whole. In various applications the term “population” can have different meanings. It represents that collective of items from which the sample is drawn. Thus it could be a shipment, a lot or a batch or any other collective entity. For the purpose of this discussion the term “population” will be used throughout.

A VASP assumes that measurements (variables) X_1, \dots, X_n for a random sample of n items from a population is available and that defectiveness for any given sample item i is equivalent to $X_i < L$, where L is some given lower specification limit. In other applications we may call item i defective when $X_i > U$, where U is some given upper specification limit. The methodology of any VASP depends on the assumed underlying distribution for the measured variables X_1, \dots, X_n . Here we will again assume that we deal with a random sample from a normal population with mean μ and standard deviation σ . The following discussion will be in terms of a lower specification limit L . The corresponding procedure for an upper specification limit U will only be summarized without derivation.

If L is a lower specification limit, then

$$p = P_{\mu, \sigma}(X < L) = P_{\mu, \sigma}\left(\frac{X - \mu}{\sigma} < \frac{L - \mu}{\sigma}\right) = \Phi\left(\frac{L - \mu}{\sigma}\right)$$

represents the probability that a given individual item in the population will be defective. Here $\Phi(x)$ denotes the standard normal distribution function and $\Phi^{-1}(p)$ its inverse. p can be interpreted as the proportion of defective

items in the population. It is in the consumer's interest to keep the probability p or proportion p of defective items in the population below a tolerable value p_1 . Keeping the proportion p low is typically costly for the producer. Hence the producer will try to keep p only so low as to remain cost effective but sufficiently low as not to trigger too many costly rejections. Hence the producer will aim for keeping $p \leq p_0$, where p_0 typically is somewhat smaller than p_1 , to provide a sufficient margin between producer and consumer interest.

For normal data the standard VASP consists in computing \bar{X} and S from the obtained sample of n items and in comparing $\bar{X} - kS$ with L for an appropriately chosen constant k . If $\bar{X} - kS \geq L$, the consumer accepts the population from which the sample was drawn and otherwise it is rejected.

Before discussing the choice of k it is appropriate to define the two notions of risk for such a VASP. Due to the random nature of the sample there is some chance that the sample misrepresents the population and thus induces us to take incorrect action. The consumer's risk is the probability of accepting the population when in fact the proportion p of defectives in the population is greater than the acceptable limit p_1 . The producer's risk is the probability of rejecting the population when in fact the proportion p of defectives in the population is $\leq p_0$.

For a given VASP let $\gamma(p)$ denote the probability of acceptance as a function of the proportion of defectives in the population. This function is also known as operating characteristic or *OC*-curve of the VASP. $\gamma(p)$ can be expressed in terms of $G_{n-1, \delta}(t)$ as follows:

$$\begin{aligned} \gamma(p) &= P_{\mu, \sigma}(\bar{X} - kS \geq L) = P_{\mu, \sigma}\left(\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} + \frac{\sqrt{n}(\mu - L)}{\sigma} \geq k\sqrt{n}\frac{S}{\sigma}\right) \\ &= P_{\mu, \sigma}\left(\frac{Z + \delta}{\sqrt{V/(n-1)}} \geq k\sqrt{n}\right) = P(T_{n-1, \delta} \geq k\sqrt{n}) \end{aligned}$$

where the noncentrality parameter

$$\delta = \delta(p) = \frac{\sqrt{n}(\mu - L)}{\sigma} = -\sqrt{n} \frac{L - \mu}{\sigma} = -\sqrt{n} \Phi^{-1}(p)$$

is a decreasing function of p . Hence

$$\gamma(p) = 1 - G_{n-1, \delta(p)}(k\sqrt{n})$$

is decreasing in p .

The consumer's risk consists of the chance of accepting the population when in fact $p \geq p_1$. In order to control the consumer's risk $\gamma(p)$ has to be kept at some sufficiently small level β for $p \geq p_1$. Since $\gamma(p)$ is decreasing in p we need only insure $\gamma(p_1) = \beta$ by proper choice of k . The factor k is then found by solving the equation

$$\beta = 1 - G_{n-1, \delta(p_1)}(k\sqrt{n}) \quad (1)$$

for k . It is customary but not necessarily compelling to choose $\beta = .10$. This solves the problem as far as the consumer is concerned. It does not address the producer's risk requirements.

The producer's risk consists of the chance of rejecting the population when in fact $p \leq p_0$. Since the probability of rejecting the population is $1 - \gamma(p)$, that probability is maximal over $p \leq p_0$ at p_0 . Hence one would limit this maximal risk $1 - \gamma(p_0)$ by some value α , customarily chosen to be .05. Note that α and β must satisfy the constraint $\alpha + \beta < 1$. Thus the producer is interested in ensuring that

$$\alpha = 1 - \gamma(p_0) = G_{n-1, \delta(p_0)}(k\sqrt{n}) \quad (2)$$

Solving this for k will typically lead to a different choice from that obtained in (1) leaving us with a conflict.

This conflict can be resolved by leaving the sample size n flexible so that there are two control parameters, n and k , which can be used to satisfy the two conflicting goals. One slight problem is that n is an integer and so it may not be possible to satisfy both equations (1) and (2) exactly. What one can do instead is the following: For a given value n find $k = k(n)$ to solve (1). If that $k(n)$ also yields

$$\alpha \geq G_{n-1, \delta(p_0)}(k(n)\sqrt{n}) , \quad (3)$$

then this sample size n was possibly chosen too high and a lower value of n should be tried. If we have

$$\alpha < G_{n-1, \delta(p_0)}(k(n)\sqrt{n}) ,$$

then n was definitely chosen too small and a larger value of n should be tried next. Through iteration one can arrive at the smallest sample size n such

that $k(n)$ and n satisfy both (1) and (3). Conversely, one could try to satisfy the exact equation (2) and maintain the appropriate inequality ($\leq \beta$) in (1) by minimal choice of n . Solving the equations (1) or (2) for k is easily done with the BCS FORTRAN subroutines HSPINT (the inverse of $G_{n-1, \delta}(t)$) and HSPNCT, which evaluates $G_{n-1, \delta}(t)$ directly, in order to check whether n was chosen too small or too large. This iteration process will lead to a solution provided $p_0 < p_1$. If p_0 and p_1 are too close to each other, very large sample sizes will be required.

In the case of an upper specification limit U we accept the lot or population whenever

$$\bar{X} + kS \leq U .$$

The *OC*-curve of this VASP is again of the form

$$\gamma(p) = P_{\mu, \sigma}(\bar{X} + kS \leq U) = 1 - G_{n-1, \delta(p)}(k\sqrt{n})$$

with $\delta(p) = -\sqrt{n}\Phi^{-1}(p)$ and p denotes again the proportion of defective items in the population, i.e.,

$$p = P_{\mu, \sigma}(X > U) = \Phi\left(\frac{\mu - U}{\sigma}\right) .$$

The parameters k and n are again determined iteratively by the two requirements

$$\alpha = G_{n-1, \delta(p_0)}(k\sqrt{n})$$

and

$$\beta = 1 - G_{n-1, \delta(p_1)}(k\sqrt{n})$$

where p_0 and p_1 ($p_0 < p_1$) are the bounds on p targeted by the producer and consumer, respectively. α and β represent the corresponding risks of the VASP, usually set at .05 and .10, respectively.

5. Tolerance Bounds

Tolerance bounds are lower or upper confidence bounds on percentiles of a population, here assumed to be normal. The discussion will mainly focus on lower confidence bounds. The upper bounds fall out immediately from the lower bounds by a simple switch to the complementary confidence level as explained below.

The p -percentile x_p of a normal population with mean μ and standard deviation σ can be expressed as

$$x_p = \mu + z_p \sigma ,$$

where $z_p = \Phi^{-1}(p)$ is the p -percentile of the standard normal distribution. The problem in bounding x_p stems from the fact that the two parameters μ and σ are unknown and will need to be estimated by \bar{X} and S . These are computed from a sample X_1, \dots, X_n taken from this population. The lower confidence bound for x_p is then computed as $\bar{X} - kS$ where k is determined to achieve the desired confidence level γ , namely so that for all (μ, σ)

$$P_{\mu, \sigma}(\bar{X} - kS \leq x_p) = \gamma .$$

By complementation this yields immediately that for all (μ, σ)

$$P_{\mu, \sigma}(\bar{X} - kS \geq x_p) = 1 - \gamma ,$$

i.e., $\bar{X} - kS$ also serves as an upper bound for x_p with confidence level $1 - \gamma$. Of course, to get a confidence level of .95 for such an upper bound one would choose $\gamma = .05$ in the above interpretation of $\bar{X} - kS$ as upper bound.

The determination of the factor k proceeds as follows:

$$\begin{aligned} P_{\mu, \sigma}(\bar{X} - kS \leq x_p) &= P_{\mu, \sigma}(\bar{X} - x_p \leq kS) = P_{\mu, \sigma}(\bar{X} - \mu - \sigma z_p \leq kS) \\ &= P_{\mu, \sigma} \left(\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} - \sqrt{n}z_p \leq \sqrt{nk} \frac{S}{\sigma} \right) = P \left(\frac{Z - \sqrt{n}z_p}{\sqrt{V/(n-1)}} \leq \sqrt{nk} \right) \\ &= P(T_{n-1, \delta} \leq \sqrt{nk}) = G_{n-1, \delta}(\sqrt{nk}) , \end{aligned}$$

where $\delta = -\sqrt{n}z_p$. Hence k is determined by solving the following equation for k :

$$G_{n-1, \delta}(\sqrt{nk}) = \gamma .$$

This is accomplished by using the BCSLIB FORTRAN subroutine HSPINT, which is the inverse of the noncentral t -distribution function $G_{f, \delta}(t)$.

6. Tail Probability Confidence Bounds

Of interest here are the tail probabilities of a normal population with mean μ and standard deviation σ . For a given threshold value x_0 one is interested in the tail probability

$$p = P_{\mu, \sigma}(X \leq x_0) = \Phi\left(\frac{x_0 - \mu}{\sigma}\right).$$

If \hat{p}_u denotes an upper bound for p with confidence level γ , i.e., for all (μ, σ)

$$P_{\mu, \sigma}(\hat{p}_u \geq p) = \gamma,$$

then we also have for all (μ, σ)

$$P_{\mu, \sigma}(\hat{p}_u \leq p) = 1 - \gamma,$$

so that \hat{p}_u can also serve as a lower bound for p with confidence level $1 - \gamma$. If the upper tail probability $1 - p$ of the normal distribution is of interest, then $1 - \hat{p}_u$ will serve as an upper bound for $1 - p$ with confidence level $1 - \gamma$. Thus it suffices to limit the discussion to upper confidence bounds for p .

In deriving these upper bounds use will be made of the following result, which is stated here in a simplified fashion:

Lemma: If X is a random variable with continuous, strictly increasing distribution function $F(t) = P(X \leq t)$, then the random variable $U = F(X)$ has a uniform distribution, i.e., $P(U \leq u) = u$ for $0 \leq u \leq 1$.

The proof of the lemma in this form is easy enough to give here:

$$P(U \leq u) = P(F(X) \leq u) = P(X \leq F^{-1}(u)) = F(F^{-1}(u)) = u.$$

As a start for constructing upper bounds for p consider

$$\frac{\sqrt{n}(x_0 - \bar{X})}{S} = \frac{\sqrt{n}(x_0 - \mu)/\sigma + \sqrt{n}(\mu - \bar{X})/\sigma}{S/\sigma} = T_{n-1, \delta},$$

and note that $Z' = \sqrt{n}(\mu - \bar{X})/\sigma$ and $Z = \sqrt{n}(\bar{X} - \mu)/\sigma = -Z'$ have the same standard normal distribution. Here $\delta = \sqrt{n}(x_0 - \mu)/\sigma = \sqrt{n}\Phi^{-1}(p)$ is an increasing function of p . By the above Lemma the random variable

$$U = G_{n-1, \delta}\left(\frac{\sqrt{n}(x_0 - \bar{X})}{S}\right) = G_{n-1, \delta}(T_{n-1, \delta})$$

has a uniform distribution over the interval $(0, 1)$ and thus it follows that

$$\gamma = P(U \geq 1 - \gamma) .$$

Since $G_{n-1, \delta}(t)$ is decreasing in δ we have

$$U \geq 1 - \gamma \quad \text{if and only if} \quad G_{n-1, \delta} \left(\sqrt{n}(x_0 - \bar{X})/S \right) \geq 1 - \gamma ,$$

$$\text{which is equivalent to} \quad \delta \leq \hat{\delta} ,$$

where $\hat{\delta}$ solves

$$G_{n-1, \hat{\delta}} \left(\sqrt{n}(x_0 - \bar{X})/S \right) = 1 - \gamma . \quad (4)$$

Hence $\hat{\delta}$ is an upper confidence bound for $\delta = \sqrt{n}\Phi^{-1}(p)$ with confidence level γ . Since

$$\hat{\delta} \geq \delta = \sqrt{n}\Phi^{-1}(p) \quad \text{if and only if} \quad \hat{p}_u \stackrel{\text{def}}{=} \Phi(\hat{\delta}/\sqrt{n}) \geq p ,$$

\hat{p}_u is the desired upper confidence bound for p with confidence level γ .

There is at this point no BCSLIB subroutine that solves equation (4) directly for $\hat{\delta}$. However, it is a simple matter to construct one, using the BCSLIB FORTRAN subroutine HSPNCT (which evaluates $G_{f, \delta}(t)$) in conjunction with HSROOT as a root finder. The latter allows for passing of additional arguments with the function whose root is to be found.

7. Bounds for Process Control Parameters C_L , C_U and C_{pk} .

Lower Specification Limits (Bounds for C_L): Let X_1, \dots, X_n be a random sample from a normal population with mean μ and standard deviation σ . Let

$$C_L = \frac{\mu - x_L}{3\sigma} ,$$

where x_L is a given lower specification limit. Denote by

$$\hat{C}_L = \frac{\bar{X} - x_L}{3S}$$

the natural estimate of C_L . The objective is to find $100\gamma\%$ lower confidence limits for C_L based on \hat{C}_L .

Similarly as in Section 4 we obtain

$$\begin{aligned} P(\hat{C}_L \leq k) &= P\left(\frac{\bar{X} - x_L}{3S} \leq k\right) \\ &= P\left(\frac{\sqrt{n}(\bar{X} - \mu)/\sigma + \sqrt{n}(\mu - x_L)/\sigma}{S/\sigma} \leq 3\sqrt{nk}\right) = P(T_{n-1,3\sqrt{n}C_L} \leq 3\sqrt{nk}). \end{aligned}$$

We define $k = k(C_L)$ as that unique number which solves

$$P(T_{n-1,3\sqrt{n}C_L} \leq 3\sqrt{nk}(C_L)) = \gamma, \text{ i.e., } P(\hat{C}_L \leq k(C_L)) = \gamma$$

and note that $k(C_L)$ is an increasing function of C_L . As lower confidence bound for C_L we take

$$\hat{B}_L = k^{-1}(\hat{C}_L)$$

and observe that

$$P(\hat{B}_L \leq C_L) = P(\hat{C}_L \leq k(C_L)) = \gamma,$$

i.e., \hat{B}_L is indeed a $100\gamma\%$ lowerbound for C_L . It remains to show how \hat{B}_L is actually computed for each observed value \hat{c}_L of \hat{C}_L .

In the defining equation for $k(C_L)$ take $C_L = k^{-1}(\hat{c}_L)$ and rewrite that defining equation as follows:

$$P(T_{n-1,3\sqrt{n}k^{-1}(\hat{c}_L)} \leq 3\sqrt{nk}(k^{-1}(\hat{c}_L))) = \gamma$$

or

$$P(T_{n-1,3\sqrt{n}k^{-1}(\hat{c}_L)} \leq 3\sqrt{n}\hat{c}_L) = \gamma.$$

If, for fixed \hat{c}_L , we solve the equation:

$$P(T_{n-1,\hat{\delta}} \leq 3\sqrt{n}\hat{c}_L) = \gamma$$

for $\hat{\delta}$, then we get the following expression for \hat{B}_L :

$$\hat{B}_L = k^{-1}(\hat{c}_L) = \frac{\hat{\delta}}{3\sqrt{n}}.$$

Upper Specification Limits (Bounds for C_U): In a similar fashion we develop lower confidence bounds for

$$C_U = \frac{x_U - \mu}{3\sigma},$$

where x_U is a given upper specification limit. Again consider the natural estimate

$$\hat{C}_U = \frac{x_U - \bar{X}}{3S}$$

of C_U . For given C_U let $k(C_U)$ be such that

$$P(\hat{C}_U \leq k(C_U)) = P(T_{n-1, 3\sqrt{n}C_U} \leq 3\sqrt{n}k(C_U)) = \gamma.$$

As before it follows that $\hat{B}_U = k^{-1}(\hat{C}_U)$ serves as 100 γ % lower confidence bound for C_U . For an observed value \hat{c}_U of \hat{C}_U we compute \hat{B}_U as $\hat{\delta}/(3\sqrt{n})$, where $\hat{\delta}$ solves

$$P(T_{n-1, \hat{\delta}} \leq 3\sqrt{n} \hat{c}_U) = \gamma.$$

Two-Sided Specification Limits (Bounds for C_{pk}): Putting the bounds on C_U and C_L together, we can obtain (slightly conservative) confidence bounds for the two-sided statistical process control parameter

$$C_{pk} = \min(C_L, C_U)$$

by simply taking

$$\hat{B} = \min(\hat{B}_L, \hat{B}_U).$$

If $C_L \leq C_U$, i.e., $C_{pk} = C_L$, then

$$\begin{aligned} P(\min(\hat{B}_L, \hat{B}_U) \leq \min(C_L, C_U)) &= P(\min(\hat{B}_L, \hat{B}_U) \leq C_L) \\ &\geq P(\hat{B}_L \leq C_L) = \gamma \end{aligned}$$

and if $C_U \leq C_L$, i.e., $C_{pk} = C_U$, then

$$\begin{aligned} P(\min(\hat{B}_L, \hat{B}_U) \leq \min(C_L, C_U)) &= P(\min(\hat{B}_L, \hat{B}_U) \leq C_U) \\ &\geq P(\hat{B}_U \leq C_U) = \gamma. \end{aligned}$$

Hence \hat{B} can be taken as lower bound for C_{pk} with confidence level at least γ . The exact confidence level of \hat{B} is somewhat higher than γ for $C_L = C_U$, i.e., when μ is the midpoint of the specification interval. As μ moves away from this midpoint the actual confidence level of \hat{B} gets very close to γ .

8. Coefficient of Variation Confidence Bounds

The coefficient of variation is traditionally defined as the ratio of standard deviation to mean, i.e., as $\nu = \sigma/\mu$. We will instead give confidence bounds for its reciprocal $\rho = 1/\nu = \mu/\sigma$. The reason for this is that \bar{X} , in the natural estimate S/\bar{X} for ν , could be zero causing certain problems. If the coefficient of variation is sufficiently small, usually the desired situation, then the distinction between it and its reciprocal is somewhat immaterial since typical bounds for ν can be inverted to bounds for ρ and vice versa. This situation is easily recognized by the sign of the upper or lower bound, respectively. If $\hat{\rho}$ as lower bound for ρ is positive, then $\hat{\nu} = 1/\hat{\rho}$ is an upper bound for a positive value of ν . If $\hat{\rho}$ as upper bound for ρ is negative, then $\hat{\nu} = 1/\hat{\rho}$ is a lower bound for a negative value of ν . In either case ρ is bounded away from zero which implies that the reciprocal $\nu = 1/\rho$ is bounded. On the other hand, if $\hat{\rho}$ as lower bound for ρ is negative, then ρ is not bounded away from zero and the reciprocal values could be arbitrarily large. Hence in that case $\hat{\nu} = 1/\hat{\rho}$ is useless as an upper bound for ν since no finite upper bound on the values of ν can be derived from $\hat{\rho}$.

To construct a lower confidence bound for $\rho = \mu/\sigma$ consider

$$\sqrt{n}\frac{\bar{X}}{S} = \frac{\sqrt{n}(\bar{X} - \mu)/\sigma + \sqrt{n}\mu/\sigma}{S/\sigma} = T_{n-1, \delta}$$

with $\delta = \sqrt{n}\mu/\sigma$. Again the random variable

$$U = G_{n-1, \delta}(\sqrt{n}\bar{X}/S) = G_{n-1, \delta}(T_{n-1, \delta})$$

is distributed uniformly over $(0, 1)$. Hence $P(U \leq \gamma) = \gamma$ so that

$$G_{n-1, \delta}(\sqrt{n}\bar{X}/S) \leq \gamma \quad \text{if and only if} \quad \hat{\delta} \leq \delta,$$

where $\hat{\delta}$ is the solution of

$$G_{n-1, \hat{\delta}}(\sqrt{n}\bar{X}/S) = \gamma \tag{5}$$

and $\hat{\rho} \stackrel{\text{def}}{=} \hat{\delta}/\sqrt{n}$ can thus be used as lower confidence bound for $\rho = \delta/\sqrt{n} = \mu/\sigma$ with confidence level γ .

To obtain an upper bound for ρ with confidence level γ one finds $\hat{\delta}$ as solution of

$$G_{n-1, \hat{\delta}}(\sqrt{n} \bar{X}/S) = 1 - \gamma \quad (6)$$

and uses $\hat{\rho} := \hat{\delta}/\sqrt{n}$ as upper bound for $\rho = \delta/\sqrt{n} = \mu/\sigma$.

Solving equations (5) and (6) proceeds along the same lines as in equation (4).

References.

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- [5] Odeh, R.E. and Owen, D.B. Owen (1980), "Tables for normal tolerance limits, sampling plans, and screening," *Marcel Dekker*, New York.
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APPENDIX A: Subprogram Usage

HSPNCT and HSPINT are available in the current release of Fortran of BCSLIB

The usage documentation in this appendix refers to other sections. These are references to the corresponding chapters of BCSLIB—not this document. These usage documentation pages are exact copies from the BCSLIB documentation.

HSPNCT: Noncentral t -Distribution Function

VERSIONS

HSPNCT — REAL

PURPOSE

HSPNCT computes the REAL probability of obtaining a random variable having a value less than or equal to x from a population with a noncentral t -distribution, with given noncentrality parameter and degrees of freedom.

RELATED SUBPROGRAMS

HSPINT Inverse of Noncentral t -Distribution Function

METHOD

If the random variables Z and V are independently distributed with Z being normally distributed with mean δ and variance 1, and V being chi-square with n degrees of freedom, then the ratio

$$X = \frac{Z}{\sqrt{V/n}}$$

has the noncentral t -distribution with n degrees of freedom and noncentrality parameter δ .

The probability of obtaining a random variable X having a value less than or equal to x (that is, the cumulative probability) from a noncentral t -distribution can be expressed as

$$P(X \leq x) = \frac{1}{\Gamma(\frac{n}{2})2^{(n/2)-1}} \int_0^\infty \Phi\left(\frac{xu}{\sqrt{n}} - \delta\right) e^{-u^2/2} u^{n-1} du,$$

where $\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^u e^{-x^2/2} dx$, which is the standardized normal probability integral.

The algorithm used is based on AS-5 published in the Journal of the Royal Statistical Society, Series C (1968), Vol. 17, No. 2.

USAGE

REAL PARM(2)

P = HSPNCT(XR, PARM, IER)

ARGUMENTS

XR	[INPUT, REAL] The value of x .
PARM	[INPUT, REAL, ARRAY] REAL array of length 2 as follows: PARM(1) The noncentrality parameter δ . PARM(2) The degrees of freedom n . PARM(2) ≥ 1 , and it must be an integer valued variable.
IER	[OUTPUT, INTEGER] Success/error code ¹ . Results have not been computed for IER < 0; HSPNCT has set P = HSMCON(1). See Section 2.2 for HSMCON. IER=0 Success, P computed. IER=-1 PARM(2) < 1. IER=-2 PARM(2) not an integral value. IER=-3 Unexpected error—see Section 1.4.2 for an explanation.
P	[OUTPUT, REAL] The desired probability.

EXAMPLE

HSPNCT may be used to compute the probability of obtaining a variable having a value less than or equal to X from a population with a noncentral t -distribution with a noncentrality parameter 0.813 and three degrees of freedom.

SAMPLE PROGRAM

```

PROGRAM SAMPLE

INTEGER    IER
REAL      P, XR, PARM(2)

REAL      HSPNCT
EXTERNAL  HSPNCT

C      Set parm for degrees of freedom to 3
XR      = 4.0
PARM(1) = 0.813
PARM(2) = 3.

```

¹ See Section 1.4.2 for a discussion of error handling.

HSPNCT

```
C    Find the probability
      P = HSPNCT( XR, PARM, IER )

      WRITE (*,9000) P, IER

      STOP
9000 FORMAT (1X, 'The probability is : ',F10.6,/
1         1X, 'IER           : ',I10 ,/)
      END
```

OUTPUT FROM SAMPLE PROGRAM

```
The probability is :   0.950000
IER                :           0
```

HSPINT: Inverse of Noncentral t -Distribution Function

VERSIONS

HSPINT — REAL

PURPOSE

HSPINT computes the REAL inverse of the cumulative probability function for the noncentral t -distribution, with n degrees of freedom and noncentrality parameter δ .

RELATED SUBPROGRAMS

HSPNCT Noncentral t -Distribution Function

METHOD

If the random variables Z and V are independently distributed with Z being normally distributed with mean δ and variance 1, and V being chi-square with n degrees of freedom, then the ratio

$$X = \frac{Z}{\sqrt{V/n}}$$

has the noncentral t distribution with n degrees of freedom and noncentrality parameter δ .

The probability of obtaining a random variable X having a value less than or equal to x (that is, the cumulative probability) from a noncentral t -distribution can be expressed as

$$P(X \leq x) = \frac{1}{\Gamma(\frac{n}{2})2^{(n/2)-1}} \int_0^\infty \Phi\left(\frac{xu}{\sqrt{n}} - \delta\right) e^{-u^2/2} u^{n-1} du,$$

where $\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^u e^{-x^2/2} dx$, which is the standardized normal probability integral.

The zero-finding program HSROOT is used to determine x where $P = P(X \leq x)$, n , and δ are given.

USAGE

REAL PARM(2)

XR = HSPINT(P, PARM, IER)

ARGUMENTS

- P** [INPUT, REAL]
Cumulative probability; $0 < P < 1$. If P is too close to 0 or 1, machine precision limitations may prevent accurate computation. If $P = 0$, then $x = 0$; if $P = 1$, then $x = \infty$.
- PARM** [INPUT, REAL, ARRAY]
Array of length 2 as follows:
PARM(1)= δ the noncentrality parameter.
PARM(2)= n the number of degrees of freedom. PARM(2) ≥ 1 , and it must be integer valued variable.
- IER** [OUTPUT, INTEGER]
Success/error code¹. Results have not been computed for IER < 0 ; HSPINT has set XR = HSMCON(1). See Section 2.2 for HSMCON.
- IER=0 Success, XR computed.
IER=-1 PARM(2) < 1 .
IER=-2 PARM(2) not an integer value.
IER=-3 $P \leq 0$ or $P \geq 1$.
IER=-4 Convergence failed with the iteration at the overflow threshold, HSMCON(2).
IER=-5 P is too close to 0 or 1.
IER=-6 Unexpected error—see Section 1.4.2 for an explanation.
through
IER=-11
- XR** [OUTPUT, REAL]
Value of x .

EXAMPLE

HSPINT may be used to compute the inverse of the cumulative probability function for the non-central t -distribution with a cumulative probability of 0.95, a noncentrality parameter 0.33769295 and three degrees of freedom.

¹ See Section 1.4.2 for a discussion of error handling.

HSPINT

SAMPLE PROGRAM

```
PROGRAM SAMPLE

INTEGER      IER
REAL         P, XR, PARM(2)

REAL         HSPINT
EXTERNAL     HSPINT

C   Set PARM for degrees of freedom to 3

P           = 0.95
PARM(1)     = 0.33769295
PARM(2)     = 3.

C   Find the inverse

XR = HSPINT( P, PARM, IER )

WRITE (*,9000) XR, IER

STOP
9000 FORMAT (1X, 'The inverse is : ',F10.6,/
1          1X, 'IER           : ',I10 ,/)
END
```

OUTPUT FROM SAMPLE PROGRAM

```
The inverse is : 3.000000
IER           : 0
```