A Strong Invariance Theorem for the Strong Law of Large Numbers

Jon A. Wellner


Stable URL: http://links.jstor.org/sici?sici=0091-1798%28197808%2963%3A4%3C673%3AASITFL%3E2.0.CO%3B2-2

*The Annals of Probability* is currently published by Institute of Mathematical Statistics.

______________________________________________________________

Your use of the JSTOR archive indicates your acceptance of JSTOR’s Terms and Conditions of Use, available at http://www.jstor.org/about/terms.html. JSTOR’s Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/journals/ims.html.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

______________________________________________________________

JSTOR is an independent not-for-profit organization dedicated to creating and preserving a digital archive of scholarly journals. For more information regarding JSTOR, please contact support@jstor.org.
A STRONG INVARiATION THEOREM FOR THE STRONG LAW OF LARGE NUMBERS

BY Jon A. WELLNER

University of Rochester

Let $X_1, X_2, \ldots$ be i.i.d. random variables with mean 0 and variance 1. Let $S_n = X_1 + \cdots + X_n$, and let $\{H_n\}$ be the standard partial sum processes on $[0, \infty)$ defined in terms of the $S_n$'s and normalized as in Strassen. Each function of the “tail” behavior of the process $H_n$ is the dual of a function of the “initial” behavior of the process $H_n$, the duality being induced by the time inversion map $R$. The dual role of “initial” and “tail” functions is used to exploit an extension of Strassen's invariance theorem for the law of the iterated logarithm due to Wichura, and thereby obtain limit theorems for a variety of functions of the “tail” behavior of the sums $S_n$. For example, with probability one,

$$\lim \sup_{n \to \infty} (n/2 \log \log n)^{\frac{1}{2}} \max_{n \leq k < \infty} (k^{-1}S_k) = 1$$

and

$$\lim \sup_{n \to \infty} n^{-1} \max \{k \geq 1 : k^{-1}S_k \geq \theta(2 \log \log n/n)^{\frac{1}{2}}\} = \theta^{-2}.$$

1. Introduction. Let $X_1, X_2, \ldots$ be i.i.d. rv’s with mean 0 defined on a common probability space $(\Omega, \mathcal{F}, P)$. Let $S_n = X_1 + \cdots + X_n$; the well-known strong law of large numbers then asserts that with probability one (w.p. 1)

$$\lim_{n \to \infty} n^{-1}S_n = 0.$$

If the $X_i$’s have finite variance (1 without loss of generality), the law of the iterated logarithm (Hartman and Wintner (1941), Strassen (1964)) yields a rate for the convergence in (1): w.p. 1 $n^{-1}S_n = O((n^{-1} \log \log n)^{\frac{1}{2}})$, or, more precisely,

$$\lim \sup_{n \to \infty} (n/2 \log \log n)^{\frac{1}{2}}(n^{-1}S_n) = 1 \quad \text{w.p. 1}.$$

But the strong law (1) also has a variety other consequences; for example

$$\lim_{n \to \infty} \max_{n \leq k < \infty} (k^{-1}S_k) = 0 \quad \text{w.p. 1},$$

and, for any $\varepsilon > 0$

$$\max \{k \geq 1 : k^{-1}S_k \geq \varepsilon\} < \infty \quad \text{w.p. 1}.$$

Our object in this note is to show that when the $X_i$’s have variance 1 then (3) and (4) can be strengthened in much the same way that (2) strengthens (1). To do this we make use of an extension of Strassen’s (1964) invariance principle for the law of the iterated logarithm which is due to Wichura (1974). We then exploit this theorem by way of a duality relationship induced by time inversion to obtain iterated logarithm type limit theorems for a variety of “tail” functions.

Received March 24, 1976; revised October 13, 1977.

1 Supported in part by the National Science Foundation under MCS 77-02255.

AMS 1970 subject classifications. Primary 60F15; Secondary 60B10.

Key words and phrases. Law of the iterated logarithm, strong law of large numbers, time inversion, invariance theorems.

673
of the $S_n$'s. Strengthened versions of (3) and (4) emerge as special cases of this general duality in Examples 2 and 3 in Section 3.

2. The main results. Define functions $H_n$, $n \geq 3$, on $[0, \infty)$ in terms of the sums $S_n$ by

$$H_n(t) = (S_{[nt]} + (nt - [nt])X_{[nt]+1})/(2n \log \log n)^{\frac{1}{t}}$$

for $t \geq 0$. $H_n$ takes values in $C[0, \infty)$, the space of continuous functions on $[0, \infty)$. Let $p_0(t) = (t \vee 1)$, $p_1(t) = ((t \vee 3) \log \log (t \vee 3))^{\frac{1}{t}}$ for $t \geq 0$, and set

$$B_i = \{x \in C[0, \infty): \sup_{0 \leq t < \infty} |x(t)|/p_i(t) < \infty\}$$

$i = 0, 1$. Define the metric $\rho_i$ for functions $x, y \in B_i$ by

$$\rho_i(x, y) = \sup_{0 \leq t < \infty} |x(t) - y(t)|/p_i(t)$$

$i = 0, 1$.

Let

$$K = \{x \in C[0, \infty): x(0) = 0, \int_0^t \hat{x}(s)\, ds, \int_0^t \hat{x}(s)^2\, ds \leq 1\}.$$

THEOREM 1 (Strassen–Wichura). With probability one the sequence $\{H_n\}_{n \geq 1}$ is relatively compact in the topology induced by the metric $\rho_0$ on $C[0, \infty)$ and has limit set $K$.

For the (easy) proof of Theorem 1 as a consequence of Strassen's (1964) invariance theorem see Wichura (1974). Our primary interest here is in the metric $\rho_0$; since $p_0 \geq p_1$ (for all $t \geq 0$) and hence $\rho_0(x, y) \leq \rho_1(x, y)$ for all $x, y \in B_1$, Theorem 1 implies that we also have convergence with respect to $\rho_0$:

COROLLARY 1. With probability one the sequence $\{H_n\}_{n \geq 1}$ is relatively compact in the topology induced by the metric $\rho_0$ on $C[0, \infty)$ and has limit set $K$.

Before proceeding to the consequences of Theorem 1 and Corollary 1 we should remark that similar theorems hold for a wide variety of processes in addition to the "partial sum" processes of the preceding discussion. Many authors have considered similar processes on $[0, 1]$ under a wide range of probabilistic assumptions and have proved analogues of Strassen's (1964) original theorem for partial sums: i.e., when properly normalized the processes are, w.p. 1, relatively compact in the topology of uniform convergence on $C[0, 1]$ and have limit set $K$ (restricted to $[0, 1]$). For example, results of this type have been established for martingales and processes with stationary increments by Heyde and Scott (1973), and for sums of weakly dependent variables by Phillip and Stout (1975). But now note that Wichura's extension of Strassen's theorem proceeds by an argument which is independent of the probabilistic assumptions imposed on the summands, and hence convergence in the topology on $C[0, \infty)$ induced by $\rho_0$ follows for any processes of this type which are relatively compact in the uniform topology on $C[0, 1]$.

A different approach to the convergence question (with respect to $\rho_0$ on $[0, \infty)$) is by way of imbedding; e.g., (1.2) of page 2 of Phillip and Stout (1975) or (3.5) page 123 or (4.5) page 127 of Jain, Jogdeo, and Stout (1975). The point is
that it is no more difficult to establish convergence with respect to \( \rho_0 \) on \([0, \infty)\) than to establish uniform convergence on finite intervals. As a consequence, the following considerations apply to a wide range of processes "like" partial sum processes.

Now let \( R \) be the standard time inversion map defined by

\[
(Rx)(t) = tx(1/t) \quad \text{for} \quad 0 < t < \infty \\
= 0 \quad \text{for} \quad t = 0.
\]

Let \( C_0 = \{ x \in C[0, \infty) : x(0) = 0, \lim_{t \to \infty} t^{-1}x(t) = 0 \} \subset B_0 \). Note that \( \mathbb{K} \subset C_0 \).

The following lemma summarizes some of the useful properties of \( R \).

**Lemma.** The time inversion map \( R \) is

(a) an isometry of the metric space \((C_0, \rho_0)\); \( \rho_0(Rx, Ry) = \rho_0(x, y) \) for all \( x, y \in C_0 \);
(b) a continuous function from \((C_0, \rho_0)\) to \((C_0, \rho_0)\);
(c) its own inverse; \( R(R(x)) = x \) for all \( x \in C_0 \);
(d) \( \mathbb{K} \)-preserving; i.e., \( R(\mathbb{K}) = \mathbb{K} \).

**Proof.** Assertions (a) and (c) are easily verified, and (b) is a consequence of (a). To see that \( R \) preserves \( \mathbb{K} \), note that \( R \) preserves Brownian motion and is continuous; hence, normalizing as in Strassen's Theorem 1, applying that theorem as extended by Wichura, and considering the resulting sets of limit points yields \( R(\mathbb{K}) = \mathbb{K} \). [ ]

Suppose that \( f(x) \) (\( f: C_0 \to \mathbb{R}^1 \)) is some measure of the "initial" behavior of functions \( x \) in \( C_0 \). Then the dual function, \( Df(x) \), defined by

\[
(Df)(x) = f(R(x)), \quad x \in C_0
\]

will be a corresponding measure of the "tail" behavior of \( x \) (and vice versa). The examples considered in Section 3 illustrate this duality between "initial" and "tail" functions.

In view of (i) the duality connection between "initial" functions and "tail" functions via the time inversion map \( R \); (ii) the fact that \( R \) is continuous and preserves \( \mathbb{K} \); and (iii) Corollary 1, it is easily seen that iterated logarithm limit theorems for "tail" functions of the processes \( H_n \) can easily be deduced from the corresponding results for "initial" functions. The following theorem makes this more precise. If a function \( f: (C_0, \rho_0) \to (\mathbb{R}^1, | |) \) is continuous at every point of \( \mathbb{B} \subset C_0 \), we say that \( f \) is \( \mathbb{B} \)-continuous on \((C_0, \rho_0)\).

**Theorem 2.** Suppose that \( f \) is a \( \mathbb{K} \)-continuous function on \((C_0, \rho_0)\) and that \( \{ H_n \}_{n \geq 1} \) is, w.p. 1, relatively compact with respect to \( \rho_0 \) with limit set \( \mathbb{K} \). Then

(a) \( Df = f \circ R \) is \( \mathbb{K} \)-continuous on \((C_0, \rho_0)\);
(b) w.p. 1 \( \{ f(H_n) \}_{n \geq 1} \) and \( \{ Df(H_n) \}_{n \geq 1} \) are relatively compact with the same limit set \( f(\mathbb{K}) = f(R(\mathbb{K})) = Df(\mathbb{K}) \); and
(c) if \( \sup_{x \in \mathbb{K}} f(x) = f(x_0), \ x_0 \in \mathbb{K}, \) then \( \sup_{x \in \mathbb{K}} Df(x) = Df(Rx_0) = f(x_0) \).

Note that if, for some subsequence \( n' \), w.p. 1 \( \{ H_n \}_{n \in (n')} \) is relatively compact
with $R$-invariant set of limit points $\mathbb{K}^* \subset \mathbb{K}$ (so $R(\mathbb{K}^*) = \mathbb{K}^*$), then (ii) and (iii) remain true with $\mathbb{K}$ replaced by $\mathbb{K}^*$; see Example 7 in this connection.

3. Examples. All of the following "initial" functions $f: C[0, 1] \rightarrow \mathbb{R}$, with the exception of Examples 3 and 6 were considered by Strassen (1964). In view of Theorem 2, we need only to translate his results for "initial" functions $Df$. We draw freely in this section on Strassen's results concerning $\sup_{x \in \mathbb{K}} f(x)$ and the $x$'s in $\mathbb{K}$ for which the supremum is obtained for the various functions $f$. Even though the functions considered in Examples 3 and 4 are not $\mathbb{K}$-continuous, one can verify that the asserted lim sup results hold by reasoning as in Example (v), pages 223–224 of Strassen (1964).

**Example 1.** Let $f_1(x) = x(s_0)$ with $0 < s_0 \leq 1$ fixed. Then $Df_1(x) = t_0^{-1}x(t_0)$ where $1 \leq t_0 = s_0^{-1} < \infty$, $\sup_{x \in \mathbb{K}} f_2(x) = s_0^{-1}$ with $x_1(t) = s_0^{-1}t \wedge s_0^1 \in \mathbb{K}$. Hence $\sup_{x \in \mathbb{K}} Df_1(x) = Df_1(Rx_1) = t_0^{-1}$ with $Rx_1(t) = t_0^{-1}t \wedge t_0^1 \in \mathbb{K}$,

$$\lim \sup_{n \rightarrow \infty} H_n(s_0) = s_0^1 \quad \text{w.p. 1},$$

and

$$\lim \sup_{n \rightarrow \infty} t_0^{-1}H_n(t_0) = t_0^{-1} \quad \text{w.p. 1};$$

further, for $n$ large $H_n(s_0)$ and $t_0^{-1}H_n(t_0)$ are close to $s_0^1 = t_0^{-1}$ iff $H_n$ is close to $x_1$ or $Rx_1$ respectively.

**Example 2.** Let $f_2(x) = \sup_{0 \leq t \leq 1} x(t)$. Then $Df_2(x) = \sup_{1 \leq t < \infty} (t^{-1}x(t))$, $\sup_{x \in \mathbb{K}} f_2(x) = 1 = f_2(x_2)$ with $x_2(t) = t \wedge 1 \in \mathbb{K}$, and $\sup_{x \in \mathbb{K}} Df_2(x) = 1 = Df_2(x_2)$ since $Rx_2 = x_2$. Note that

$$f_3(H_n) = (\max_{1 \leq k \leq n} S_k)/(2n \log \log n)^{1},$$

and

$$Df_3(H_n) = (n/2 \log \log n)^{1} \max_{1 \leq k < \infty} (k^{-1}S_k).$$

Hence Theorem 2 implies that

$$\lim \sup_{n \rightarrow \infty} (2n \log \log n)^{-1}(\max_{1 \leq k \leq n} S_k) = 1 \quad \text{w.p. 1},$$

and

$$\lim \sup_{n \rightarrow \infty} (n/2 \log \log n)^{1} \max_{1 \leq k \leq n} (k^{-1}S_k) = 1 \quad \text{w.p. 1};$$

furthermore for large $n$ $f_3(H_n)$ and $Df_3(H_n)$ are close to 1 iff $H_n$ is close to $x_2 = Rx_2$. This example strengthens (3) in the presence of a second moment.

**Example 3.** For $\theta > 0$ let $f_3(x) = (\inf \{ t \geq 0 : x(t) = \theta \})^{-1}$ where the infimum equals $+\infty$ if the set is empty (and hence $f_3(x) = 0$). Then $Df_3(x) = \sup \{ t \geq 0 : t^{-1}x(t) = \theta \}$, $\sup_{x \in \mathbb{K}} f_3(x) = \theta^{-2} = f_3(x_2)$ with $x_2(t) = \theta^{-1}t \wedge \theta \in \mathbb{K}$, and $\sup_{x \in \mathbb{K}} Df_3(x) = \theta^{-2} = Df_3(Rx_2)$ with $Rx_2(t) = \theta t \wedge t^{-1}$. Note that

$$P(\lim \sup_{n \rightarrow \infty} Df_3(H_n) = \lim \sup_{n \rightarrow \infty} n^{-1} \max \{ m \geq 1 : m^{-1}S_m \geq \theta (2 \log \log n)^{1} \}) = 1$$

and hence

$$\lim \sup_{n \rightarrow \infty} n^{-1} \max \{ m \geq 1 : m^{-1}S_m \geq \theta (2 \log \log n)^{1} \} = \theta^{-2} \quad \text{w.p. 1}.$$
furthermore, for large \( n \) \( Df_n(H_n) \) is close to \( \theta^{-2} \) iff \( H_n \) is close to \( Rx_\theta(t) = \theta t \wedge \theta^{-1} \). This example strengthens (4) in the presence of a finite second moment.

**Example 4.** For \( 0 < c < 1 \) let \( f_c(x) = \lambda[0 \leq t \leq 1: x(t) \geq ct^c] = \int_0^1 1_{(c^t; t^c \leq ct^c)} \, dt \) where \( \lambda \) denotes Lebesgue measure. Then \( Df_c(x) = \int_0^\infty \frac{t^{1-c^t}}{(1-t)/(2e)} \frac{t^{-c}}{(-1)^{c-1}} s^{-2} \, ds \), \( \sup_{x \in K} f_c(x) = 1 - \exp(-4(c-2) - 1) \equiv 1 - s_c = f_c(x_0) \) where \( x_0(t) = \sigma^{-1} s^{-1}(s \wedge t) \sigma(s) \, ds \), \( \sigma^{-1} \int_0^1 \sigma(s) \, ds \) according as \( 0 \leq t < \sigma, \sigma \leq t \leq 1, 0 \leq t < \infty \) and \( Rx_\sigma(t) = ct, ct^c, \sigma^{-1}s^{-1} \) according as \( 0 \leq t \leq 1, 1 \leq t \leq \sigma^{-1}, 0 \leq t < \infty \). Hence

\[
\lim \sup_{n \to \infty} Df_n(H_n) = 1 - \exp(-4(c-2) - 1) \quad \text{w.p. 1}.
\]

**Example 5.** If \( \phi \) is a fixed Riemann integrable real function on \([0, 1]\) let \( f_\phi(x) = \int_0^1 x(t) \phi(t) \, dt \). Then \( Df_\phi(x) = \int_0^\infty t^{-1}x(t) \phi(t) t^{-2} \, dt \) where \( \phi(t) = \phi(1/t) \) for \( 1 \leq t < \infty \), \( \sup_{x \in K} f_\phi(x) = \int_0^1 \Phi(t) \, dt \equiv \sigma = f_\phi(x_0) \) where \( x_0(t) = \sigma^{-1} s^{-1}(s \wedge t) \phi(s) \, ds \), \( \sigma^{-1} \int_0^1 \phi(s) \, ds \) according as \( 0 \leq t \leq 1 \) or \( 1 \leq t < \infty \), and \( \Phi(t) = \int_0^1 \phi(s) \, ds \). Thus \( Rx_\phi(t) = t^{-\sigma} \int_0^\infty s^{-\sigma} \phi(s) \, ds \), \( \sigma^{-1} \int_0^1 s^{-1}(s \wedge t) \phi(s) s^{-2} \, ds \) according as \( 0 \leq t \leq 1 \), or \( 1 \leq t < \infty \). Hence it follows from Theorem 2 that

\[
\lim \sup_{n \to \infty} Df_n(H_n) = \sigma \quad \text{w.p. 1} ;
\]

furthermore, for large \( n \) \( Df_n(H_n) \) is close to \( \sigma \) iff \( H_n \) is close to \( Rx_\sigma \).

**Example 6.** Let \( f_{a}(x) = \int_0^1 t^{-1}x(t) \, dt - x(1) \). Then \( Df_{a}(x) = \int_0^\infty t^{-1}x(t) \, dt - x(1) = \int_0^\infty t^{-1} \, dx(t) \), \( \sup_{x \in K} f_{a}(x) = f_{a}(x_0) \) with \( x_0(t) = t \log(1/t) \), 0 according as \( 0 \leq t \leq 1 \) or \( 1 \leq t < \infty \); and \( \sup_{x \in K} Df_{a}(x) = Df_{a}(Rx_\sigma) = 1 \) with \( Rx_\sigma(t) = 0 \), \( \log(t) \) according as \( 0 \leq t \leq 1 \) or \( 1 \leq t < \infty \). Note that

\[
Df_{a}(H_n) = (n/2 \log \log n)^{1/2} \sum_{i=1}^{n+1} X_i \left(- \log \left(1 - \frac{1}{i} \right)\right),
\]

and hence it follows from Theorem 2 that

\[
\lim \sup_{n \to \infty} (n/2 \log \log n)^{1/2} \sum_{i=1}^{n+1} X_i \left(- \log \left(1 - \frac{1}{i} \right)\right) = 1 \quad \text{w.p. 1} ;
\]

and, for large \( n \) \( Df_n(H_n) \) is close to 1 iff \( H_n \) is close to \( Rx_\sigma(t) = \log(t) \) on \([1, \infty)\). This example is related to a type of duality studied by Barbour (1974); see the discussion in the following section in this connection.

**Example 7.** This final example illustrates the remark following Theorem 2 concerning \( R \)-invariant subsets of \( K \). Fix \( \beta \in [-1, +1] \); for every \( \omega \) in a set with probability one there is a subsequence \( \{n'\} = \{n'(\omega)\} \) such that \( \lim_{n \to \infty} H_n(1) = \beta \); further, w.p. 1, the functions \( \{H_n\}_{n \in \mathbb{N}'} \) are relatively compact with respect to \( \rho_0 \) and have limit set \( K_\beta \equiv \{x \in K: x(1) = \beta\} \). Note that \( R(K_\beta) = K_{\beta'} \); i.e., \( K_\beta \) is \( R \)-invariant. Let \( f_\beta(x) = \sup_{0s \leq t \leq 1} x(t) \) as in Example 2. Then \( Df_{\beta}(x) = \sup_{0s \leq t \leq 1} (t^{-1}x(t)) \) as before, but now we have \( \sup_{x \in K} f_{\beta}(x) = \sup_{x \in K, x(1)=\beta} \sup_{0s \leq t \leq 1} x(t) = 3/4 + \beta = f_{\beta}(x_0) \) with \( x_0(t) = t, 1 + \beta - t, \beta \) according as \( 0 \leq t \leq (1 + \beta)/2, (1 + \beta)/2 \leq t \leq 1 \), or \( 1 \leq t < \infty \). Hence \( \sup_{x \in K_{\beta}} Df_{\beta}(x) = 3/4 + \beta = Df_{\beta}(Rx_\beta) \) where \( Rx_\beta(t) = \beta t, (1 + \beta)t - 1, 1 \) according as \( 0 \leq t \leq 1 \),
\[ 1 \leq t \leq 2(1 + \beta)^{-1}, \text{ or } 2(1 + \beta)^{-1} \leq t < \infty, \text{ and it follows from Theorem 2 that} \]
\[
\limsup_{n \to \infty} (n/2 \log \log n)^\frac{1}{3} \max_{k \leq \log n} (k^{-1}S_k) = \frac{1}{2}(1 + \beta) \quad \text{w.p. 1.}
\]

Similar questions, involving limit superiors along subsequences and consequent restrictions to the subsets \( \mathbb{K}_\delta \) of \( \mathbb{K} \), could be raised for each of the five other examples considered above.

4. Related work. A “weak” invariance theorem for the strong law of large numbers was proved by Müller (1968); one of his theorems asserts that the processes \( (H_n) \), with the \( (2 \log \log n)^{\frac{1}{3}} \) factor omitted, converge weakly to Brownian motion \( B \) in \( (C_0, \rho_0) \). From this he deduced, via time inversion, that
\[
\lim_{n \to \infty} P(n^3 \max_{n \leq k < \infty} k^{-1}S_k \leq \lambda) = P(\sup_{1 \leq t < \infty} t^{-1}B(t) \leq \lambda)
\]
\[
= P(\sup_{0 \leq t \leq 1} B(t) \leq \lambda)
\]
\[
= (2/\pi)^{\frac{1}{3}} \int_0^\infty \exp(-u^2/2) \, du,
\]
which is the “weak” version of our Example 2; that
\[
\lim_{n \to \infty} P(\max \{ k : k^{-1}S_k \geq \theta u^{-1} \} \leq \lambda n) = P(\sup \{ t : B(t) \geq \theta t \} \leq \lambda)
\]
\[
= P(\inf \{ t : B(t) \geq \theta \} \geq \lambda^{-1})
\]
\[
= \theta^{-\frac{1}{3}} (2\pi u)^{-1} \exp(-\theta^2 u/2) \, du,
\]
which is the “weak” version of our Example 3; and that
\[
\lim_{n \to \infty} P(n^3 \max_{n \leq k < \infty} (k^{-1}S_k) \geq \alpha | n^{-1}S_n = n^{-1} \beta)
\]
\[
= P(\sup_{1 \leq t < \infty} (t^{-1}B(t)) \geq \alpha | B(1) = \beta)
\]
\[
= P(\sup_{0 \leq t \leq 1} B(t) \geq \alpha | B(1) = \beta)
\]
\[
= \exp(-2\alpha(\alpha - \beta)) \quad \text{for} \quad \alpha \geq \beta \geq 0,
\]
which is the “weak” version of our Example 7. The present note was largely inspired by Müller’s (1968) paper.

If \( Y_1, Y_2, \ldots \) are independent rv’s with mean 0 and \( \text{Var} \, Y_n = \sigma_n^2 \), then it is well known that \( \sum_{n=1}^\infty \sigma_n^2 < \infty \) implies that \( \sum_{n=1}^\infty Y_n < \infty \) w.p. 1. Barbour (1974) studied the relationship between limit theorems for the “initial” sums \( S_n = \sum_{i=1}^n Y_i \) and the “tail” sums \( T_n = \sum_{i=n}^\infty Y_i \). His results depend on a duality induced by the map \( G \) from \( C_0 \) (say) to \( C_0 \) defined by
\[
Gx(t) = tx(1/t) - \int_t^\infty u^{-2}x(u) \, du \quad t > 0.
\]

\( G \) plays much the same role in Barbour’s paper that \( R \) plays in ours; \( G \) preserves \( \mathbb{K} \) and Brownian motion, is continuous, and \( G \circ G \) is the identity. The duality theme of his paper is similar to ours, but the processes considered by Barbour are different than the processes considered here. It would be interesting to know of other \( \mathbb{K} \)-preserving mappings, and their interrelations and uses.

Acknowledgments. I wish to thank the referee for his very helpful comments and suggestions; in particular Theorem 2 and the applicability to other than
ITERATED LOGARITHM LAWS FOR THE STRONG LAW

partial sum processes were both suggested by him. I also owe thanks to F. W. and Roberta Scholz for their valuable translation of the paper by D. W. Müller.

REFERENCES


DEPARTMENT OF STATISTICS
UNIVERSITY OF ROCHESTER
ROCHESTER, NEW YORK 14627