Marginal set likelihood for semiparametric copula estimation

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October 18, 2006

Abstract

Quantitative studies in many fields involve the analysis of multivariate data of diverse types, including measurements that we may consider binary, ordinal and continuous. One approach to the analysis of such mixed data is to use a copula model, in which the associations among the variables are parameterized separately from their univariate marginal distributions. The purpose of this article is to provide a method of semiparametric inference for copula models via the use of what we call a marginal set likelihood function for the association parameters. The proposed method of inference can be viewed as a generalization of marginal likelihood estimation, in which inference for a parameter of interest is based on a summary statistic whose sampling distribution is not a function of any nuisance parameters. In the context of copula estimation, the marginal set likelihood is a function of the association parameters only and its applicability does not depend on any assumptions about the marginal distributions of the data, thus making it appropriate for the analysis of mixed continuous and discrete data with arbitrary marginal distributions. Estimation and inference for parameters of the Gaussian copula are available via a straightforward Markov chain Monte Carlo algorithm based on Gibbs sampling.

Some key words: Bayesian inference, latent variable model, marginal likelihood, Markov chain Monte Carlo, multivariate estimation, polychoric correlation, rank likelihood, sufficiency.

1 Introduction

Studies involving multivariate data often include measurements of diverse types. For example, a survey or observational study may record the sex, education level and income of its participants, thus including measurements that we may consider binary, ordinal and continuous. Many surveys also include questions about attitudes and preferences measured on Likert scales. In such cases, the interest is primarily in the association among the variables, and not the scale on which they are measured.

One approach to the analysis of such data is to use rank-based measures of bivariate association. These procedures are scale-free, but involve ad-hoc methods for dealing with ties in discrete data and provide inference that is generally limited to hypothesis tests of bivariate association. In contrast, model-based procedures can address a variety of inferential questions and can directly take into account the sample space of each measured variable. These methods generally proceed by modeling each component of a vector of observations with a parametric exponential family model, in which the parameters for each component involve an unobserved latent variable. For example, Chib and Winkelmann (2001) present a model for a vector of correlated count data in which each component is a Poisson random variable with a mean depending on a component-specific latent variable. Dependence among the count variables is induced by modeling the vector of latent variables with a multivariate normal distribution. Similar approaches are proposed by Dunson (2000) and described in Chapter 8 of Congdon (2003). The model of Chib and Winkelmann can be viewed as a copula model, in which the association parameters are modeled separately from the marginal distributions of the observed data. Such a modeling approach can be applied to a wide variety of multivariate analysis problems: An old mathematical result known as Sklar’s Theorem says that every multivariate probability distribution can be represented by its univariate marginal distributions and a copula, which is a type of joint distribution with fixed marginals.

The marginal distributions of survey data such as age, number of children, income and ordinal preferences generally do not belong to standard parametric families. For such data a semiparametric estimation strategy may be appropriate, in which the associations among the variables are represented with a simple parametric model but the marginal distributions are estimated nonparametrically. In the case where all the variables are continuous, Genest et al. (1995) suggest a “pseudo-likelihood” approach to estimation, in which the observed data is transformed via the empirical marginal distributions to obtain pseudo-data that can be used to estimate the association parameters. Klaassen and Wellner (1997) study a similar type of estimation in the case of the Gaussian copula. Such estimators are well-behaved for continuous data but can fail for discrete data, making them somewhat inappropriate for the analysis of mixed continuous and discrete data.

For ordinal discrete data with a known number of categories, the dependence induced by the Gaussian copula model is called polychoric correlation. Olsson (1979) describes a two-stage estimation procedure for the parameters in the copula, and this and other estimation strategies appear in a number of software packages including SAS PROC FREQ and the LISREL module PRELIS. Kottas et al. (2005) describe a nonparametric estimation procedure in which the copula is based on a mixture of normal distributions. However, such procedures do not accommodate continuous data, and may even be problematic for discrete data with a large number of categories, as inference in this case requires the simultaneous estimation of the large number of parameters specifying the marginal distributions.
As an alternative to these procedures, this article presents an approach to copula estimation in which the marginal distributions are arbitrary and of unspecified types, thus accommodating both discrete and continuous data. This is achieved by the use of a likelihood function that depends on the association parameters only, and does not make assumptions about the form of the univariate marginal distributions. Inference based on such a likelihood is therefore appropriate for the joint analysis of continuous and ordinal discrete data. For continuous data, the likelihood function we propose is derived from the marginal probability of the ranks, and can be seen as a multivariate version of a “rank likelihood” (Pettitt [1982], Heller and Qin [2001]) which does not depend on the univariate marginal distributions. Unfortunately, for discrete data the probability of the observed ranks is not free of these nuisance parameters. To solve this problem, we derive a likelihood that is equivalent to the distribution of the ranks for continuous data but is also free of the nuisance parameters for discrete data. This likelihood function is derived from the probability that the latent variables of the copula model satisfy the partial ordering induced by the observed data. We call this function a marginal set likelihood, as it is based on the marginal probability of a set. This can be seen as a generalization of a marginal likelihood, which is based on a statistic whose sampling distribution depends only on the parameter of interest and not on any nuisance parameters.

In what follows we work with the Gaussian copula model, although the basic ideas can be extended to other parametric families of copulas. In the next section we review the general Gaussian copula model, and discuss how inference for discrete data using existing semiparametric methods is problematic. Section 3 derives the set likelihood as a general approach to semiparametric copula estimation and discusses parameter estimation in the context of Bayesian inference using a relatively simple Gibbs sampling scheme. Section 4 gives an example analysis of data from the General Social Survey, a multivariate dataset that includes a number of discrete and non-Gaussian random variables. Section 5 considers notions of statistical sufficiency relevant to the set likelihood, and a discussion follows in Section 6. A short computer program for Gaussian copula estimation is provided in the Appendix.

2 Semiparametric copula estimation

Let $y_1$ and $y_2$ be two random variables with continuous CDF’s $F_1$ and $F_2$. The transformed variables $u_1 = F_1(y_1)$ and $u_2 = F_2(y_2)$ both have uniform marginal distributions. The term “copula modeling” generally refers to a model that parametrizes the joint distribution of $u_1$ and $u_2$ separately from the marginal distributions $F_1$ and $F_2$. A semiparametric copula model includes a parametric model for the joint distribution of $u_1$ and $u_2$, but lacks any parametric restrictions on $F_1$ or $F_2$.

Any continuous multivariate distribution can be used to form a copula model via an inverse-CDF
the usual copula formulation can be seen by noting that \( \Phi(z) \) values from a Gaussian copula. If the marginal distributions are continuous and known, then the values \( z_{i,j} = \Phi^{-1} [F_j(y_{i,j})] \) could be treated as observed data and \( \rho \) could be estimated directly from the \( z \)'s, perhaps using the unbiased estimator \( \hat{\rho} = \frac{1}{n} \sum_{i=1}^{n} z_{i,1} z_{i,2} \). Of course, the marginal CDF’s are not typically known. One semiparametric estimation strategy is to plug-in the the empirical CDF’s \( \hat{F}_1 \) and \( \hat{F}_2 \) to obtain pseudo-data \( \tilde{z}_{i,j} = \Phi^{-1} [\frac{1}{n} \hat{F}_j(y_{i,j})] \equiv \Phi^{-1} [\hat{F}_j(y_{i,j})] \), where the rescaling is to avoid infinities. For continuous data, the estimator \( \hat{\rho} = \frac{1}{n} \sum_{i=1}^{n} \tilde{z}_{i,1} \tilde{z}_{i,2} \) is asymptotically equivalent to the asymptotically efficient Van der Waerden normal-scores rank correlation coefficient (Hájek and Šidák 1967; Klaassen and Wellner 1997). This estimator is similar to one obtained from a more general pseudo-likelihood estimation procedure described and studied by Genest et al. (1995).

In the context of the Gaussian copula model, the maximum pseudo-likelihood procedure is to

1. set \( \tilde{z}_{i,j} = \Phi^{-1} [\hat{F}_j(y_{i,j})] \);

2. maximize in \( \rho \) the pseudo-log-likelihood \( \sum_{i=1}^{n} \log \text{bvn}(\tilde{z}_{i,1}, \tilde{z}_{i,2} | \rho) \),

where \( \text{bvn}(\cdot | \rho) \) denotes the bivariate normal density with standard normal marginals. Genest et al. show that the resulting pseudo-likelihood estimator is consistent and asymptotically normal under the condition that \( F_1 \) and \( F_2 \) are continuous. However, this condition calls into question the appropriateness of the pseudo-likelihood approach for non-continuous data such as sex, education level, age or any other type of data where there are likely to be ties.

What could go wrong with such an estimator in situations involving discrete data? In general, these pseudo-data estimators of copula parameters will be problematic for discrete data because transformations of such data do not really change the data distribution, they just change the sample space. Consider the simple case of a continuous variable \( y_1 \) and a binary variable \( y_2 \) such that \( \Pr(y_2 = 0) = \Pr(y_2 = 1) = 1/2 \). Letting \( \tilde{z}_{i,j} = \Phi^{-1} [\hat{F}_j(y_{i,j})] \), the distribution of \( \tilde{z}_{1,1}, \ldots, \tilde{z}_{n,1} \) will have an approximately standard normal distribution, but \( \tilde{z}_{i,2} \) will be approximately equal to either \( \Phi^{-1} (\frac{1}{2} \frac{n}{n+1}) \) or \( \Phi^{-1} (\frac{n}{n+1}) \) with probability one-half each. If the Gaussian copula model is correct, then one can show that the expectation of \( \hat{\rho} \) is roughly \( \frac{\rho}{\sqrt{2\pi}} \Phi^{-1} (\frac{n}{n+1}) \). As \( n \) increases so does the expectation of \( \hat{\rho} \), and it is not a consistent estimator. One problem here is that all of the
\( z_{i,2} \)'s such that \( y_{i,2} = 1 \) are being pushed to the extreme standard normal quantile \( \Phi^{-1}(\frac{n}{n+1}) \), which in the case of continuous data would happen just to a single datapoint. The situation is only partly improved by using the sample correlation of the pseudo-data as an estimator: The variance of \( z_1 \) is approximately 1 and the variance of \( z_2 \) is approximately \( [\frac{1}{2} \Phi^{-1}(\frac{n}{n+1})]^2 \), giving an approximate sample correlation of \( \text{Cor}(\tilde{z}_{i,1}, \tilde{z}_{i,2}) \approx \rho \sqrt{2/\pi} \).

3 Estimation using the marginal set likelihood

In this section we derive a likelihood function that depends on the association parameters and not on the unknown marginal distributions. For continuous data this function is equivalent to the distribution of the multivariate ranks. This is not the case of discrete data, for which the distribution of the ranks depends on the univariate marginal distributions. In this case the derived likelihood function contains less total information than one based on the ranks, but it is free of any parameters describing the marginal distributions.

3.1 Marginal set likelihood

Generalizing from the previous section, the Gaussian copula sampling model can be expressed as follows:

\[
\begin{align*}
  z_1, \ldots, z_n | C & \sim \text{i.i.d. multivariate normal}(0, C), \\
  y_{i,j} & = F_j^{-1}[\Phi(z_{i,j})],
\end{align*}
\]

where \( C \) is a \( p \times p \) correlation matrix and each \( F_j^{-1} \) denotes the (pseudo) inverse of an unknown univariate CDF, not necessarily continuous.

Our goal is to make inference on \( C \), and not on the potentially high-dimensional parameters \( F_1, \ldots, F_p \). If the \( z \)'s were observed we could use them to directly estimate \( C \). The \( z \)'s are not observed of course, but the \( y \)'s do provide a limited amount of information about them, even absent any knowledge of the \( F \)'s: Since the \( F \)'s are non-decreasing, observing \( y_{i1,j} < y_{i2,j} \) implies that \( z_{i1,j} < z_{i2,j} \). More generally, observing \( Y = (y_1, \ldots, y_n)^T \) tells us that \( Z = (z_1, \ldots, z_n)^T \) must lie in the set

\[
\{ Z \in \mathbb{R}^{n \times p} : \max\{z_{k,j} : y_{k,j} < y_{i,j}\} < z_{i,j} < \min\{z_{k,j} : y_{i,j} < y_{k,j}\} \}.
\]

We can take the occurrence of this event as our data. Letting \( D \) be the fixed subset of \( \mathbb{R}^{n \times p} \) generated by the observed value of \( Y \), we can calculate the following “likelihood”:

\[
\Pr(Z \in D | C, F_1, \ldots, F_p) = \int_D p(Z|C) \, dZ = \Pr(Z \in D | C).
\]

As a function of the parameters, this likelihood depends only on the parameter of interest \( C \) and not the nuisance parameters \( F_1, \ldots, F_p \). Estimation of \( C \) can proceed by maximizing \( \Pr(Z \in D | C) \)
as a function of $C$, or by obtaining a posterior distribution $Pr(C|Z \in D) \propto p(C) \times Pr(Z \in D|C)$. We call this likelihood function a marginal set likelihood, or s-likelihood, as it is based on the marginal probability of an event: Roughly speaking, we have
\[
p(Y|C, F_1, \ldots, F_p) = p(Y, Z \in D|C, F_1, \ldots, F_p) = Pr(Z \in D|C) \times p(Y|Z \in D, C, F_1, \ldots, F_p).
\]

Equation (2) holds because the event $Z \in D$ occurs whenever $Y$ is observed. This derivation can be made more rigorous by deriving the density $p(Y|C, F_1, \ldots, F_p)$ from the limit of $Pr(\cap_{i,j}(y_{i,j} - \epsilon, y_{i,j})|C, F_1, \ldots, F_p)$.

### 3.2 Estimation of the copula parameters

Bayesian inference for $C$ can be achieved via construction of a Markov chain having a stationary distribution equal to $p(C|Z \in D) \propto p(C) \times p(Z \in D|C)$. In the case of the Gaussian copula with a semi-conjugate prior distribution, the Markov chain can be constructed quite easily using Gibbs sampling. This prior distribution for $C$ is defined as follows: Let $V$ have an inverse-Wishart$(\nu_0, \nu_0V_0)$ prior distribution, parameterized so that $E[V^{-1}] = V_0^{-1}$, and let $C$ be equal in distribution to the correlation matrix with entries $V_{[i,j]}/\sqrt{V_{[i,i]}V_{[j,j]}}$. Using this prior distribution, approximate samples from $p(C|Z \in D)$ can be obtained by iterating the following Gibbs sampling scheme:

**Resample $Z$.** Iteratively over $(i, j)$, sample $z_{i,j}$ from $p(z_{i,j}|Z_{[-i,-j]}, V)$ as follows:

For each $j \in \{1, \ldots, p\}$

For each $y \in \text{unique}\{y_{1,j}, \ldots, y_{n,j}\}$

1. Compute $z_l = \max\{z_{i,j} : y_{i,j} < y\}$ and $z_u = \min\{z_{i,j} : y < y_{i,j}\}$
2. For each $i$ such that $y_{i,j} = y$,
   - (a) compute $\sigma_j^2 = V_{[i,j]} - V_{[i,-j]}V_{[-j,-j]}^{-1}V_{[-j,j]}$
   - (b) compute $\mu_{i,j} = Z_{[i,-j]}(V_{[i,-j]}V_{[-j,-j]}^{-1})^T$
   - (c) Sample $u_{i,j}$ uniformly from $(\Phi[\frac{z_l - \mu_{i,j}}{\sigma_j}], \Phi[\frac{z_u - \mu_{i,j}}{\sigma_j}])$
   - (d) Set $z_{i,j} = \mu_{i,j} + \sigma_j \times \Phi^{-1}(u_{i,j})$

**Resample $V$.** Sample $V$ from an inverse-Wishart$(\nu_0 + n, \nu_0V_0 + Z^TZ)$ distribution.

**Compute $C$.** Let $C_{[i,j]} = V_{[i,j]}/\sqrt{V_{[i,i]}V_{[j,j]}}$.

Iteration of this algorithm generates a Markov chain in $C$ whose stationary distribution is $p(C|Z \in D)$. This algorithm is easily modified to accommodate data that are missing-at-random: If $y_{i,j}$ is
missing, the full conditional distribution of \( z_{i,j} \) is the unconstrained normal distribution with mean \( \mu_{i,j} \) and variance \( \sigma_j^2 \) given above.

The astute reader may have noticed that the samples of \( Z \) are based on the covariance matrix \( V \) and not the correlation matrix \( C \). To see why this does not matter for estimation of \( C \), compare our original model,

\[
V \sim \text{inverse-Wishart}(\nu_0, \nu_0 V_0) \\
\{C[i,j]\} = \{V[i,j]/\sqrt{V[i,i]V[j,j]}\} \\
z_1, \ldots, z_n \sim \text{i.i.d. multivariate normal}(0, C) \\
y_{i,j} = G_j(z_{i,j}),
\]

to the equivalent model

\[
V \sim \text{inverse-Wishart}(\nu_0, \nu_0 V_0) \\
z_1, \ldots, z_n \sim \text{i.i.d. multivariate normal}(0, V) \\
\tilde{z}_{i,j} = z_{i,j}/\sqrt{V[j,j]}, \text{ and let } C = \text{Cov}(\tilde{z}) \\
y_{i,j} = G_j(\tilde{z}_{i,j}).
\]

The \( z \)'s in the first formulation are equal in distribution to the \( \tilde{z} \)'s in the second, and so posterior inference for \( C \) is equivalent under either model. The Gibbs sampling scheme outlined above is based on a Markov chain in \( V \) and \( z_1, \ldots, z_n \) based on the second formulation. Note that in this formulation the observed data implies the same ordering \( D \) on both the \( \tilde{z} \)'s and the \( z \)'s.

A short computer program to implement the estimation strategy outlined above is provided in the Appendix.

4 GSS Example

The General Social Survey (GSS) is currently a biannual survey of the noninstitutionalized U.S. adult population. Questions on the survey include a wide variety of discrete and (somewhat) continuous demographic data as well as data on opinions and attitudes measured on Likert scales. Data and details on the survey are available at [http://webapp.icpsr.umich.edu/GSS/](http://webapp.icpsr.umich.edu/GSS/) In this section we describe the dependence relationships among a set of 10 ordinal variables of interest to the author, using the model and estimation scheme of Section 3. The variables of interest are:
Figure 1: Markov chain Monte Carlo samples of 11 of the correlation coefficients.

SEX: sex of the respondent
AGE: age of the respondent
CHILDS: number of children ever had
EDU: number of years of education
PAREDU: maximum of mother’s and father’s years of education
WORDSUM: score on a vocabulary test
INCOME: family income
ATTEND: church attendance, from low to high
RELITEN: strength of religious belief, from not religious to strongly religious
BIBLE: belief that the bible is a book of fables, an inspired book, or the word of god

Relationships among the variables were analyzed using data from the $n = 2832$ respondents in the 1998 survey. This is the most recent survey for which the data are publicly available on the GSS webpage. The number of respondents having no missing data for these 10 variables was 667, or about 24%. Most of the variables had missing-data rates of about 10% or less. The exceptions were BIBLE, having a rate of about 20%, and WORDSUM, having a rate of about 54%. The majority of these missing data result from the fact that these two questions were not asked on all versions of the survey. These missing values can be reasonably considered missing at random.
Figure 2: Marginal distributions and dependence parameters for the GSS data. The plot in the second column gives posterior mean estimates of correlation coefficients $E[z_j z_k]$. The plot in the third column gives regression coefficients $\nabla E[z_j | z_{-j}]$. 
4.1 Estimation of C

Using an inverse-Wishart \((p + 2, (p + 2) \times I)\) prior distribution for \(V\), the Gibbs sampling scheme outlined in Section 3 was iterated 25,000 times with with parameter values saved every 20 scans, resulting in 1250 samples of \(C\) for posterior analysis. Mixing of the Markov chain was quite good: Figure 1 shows MCMC samples of 11 entries of \(C\), the entries being chosen to span the range of \(E[C|Z \in D]\). Convergence to stationarity appears to occur quickly, almost certainly within the first 5000 scans. Dropping these scans to allow for burn-in, we are left with 1000 saved scans for posterior analysis. The autocorrelation across these saved scans was low, with the lag-20 autocorrelation less than 0.05 in absolute value for all entries of \(C\), and much closer to zero for most.

The first column of Figure 2 gives a histogram of each variable, highlighting the non-normality of many of these marginal distributions. Posterior distributions of the correlation parameters are summarized in the second and third columns: The second column gives marginal posterior 2.5%, 50% and 97.5% quantiles of the correlation coefficients, and the third column gives the same quantiles for the “regression coefficients” \(C_{[j,-j]}C_{[-j,-j]}^{-1}\). The confidence intervals are quite small due to the large sample size. Various interesting relationships can be determined from these coefficients. For example, INCOME and PAREDU are positively correlated, but their association conditional on EDU (and the other variables) is much weaker.

4.2 Asymptotic experiments

As discussed in the next section, the s-likelihood for discrete data does not correspond to the marginal distribution of a sufficient statistic, and so estimation of the correlation parameters using the s-likelihood may differ from estimation using the full likelihood. To explore potential differences, we first obtained the Bayes estimate of \(C\) using the full likelihood \(p(Y|C, F_1, \ldots, F_p)\) and the same prior distribution for \(C\) as above. In this case, Bayesian estimation requires that prior distributions be specified for each of the \(p = 10\) univariate marginal distributions. Although some of the 10 variables have a large number of levels (AGE has 72, INCOME has 23), each variable is technically discrete as a result of how the data were recorded. Letting \(K_j\) be the number of possible levels of variable \(j\), the prior distribution for each \(F_j\) was taken to be the “standard” default prior distribution for a vector of multinomial probabilities, the uniform distribution on the \(K_j\)-simplex.

Note that if any of the marginal distributions were truly continuous then semiparametric inference via the full likelihood would require a much more elaborate set of prior distributions, such as a Pólya tree prior \([\text{Lavine, 1992}]\), for each continuous marginal. Estimation via the s-likelihood involves no such complication.

A Markov chain consisting of 25,000 scans was constructed using Gibbs sampling on all parameters, with the first 5,000 scans being discarded to allow for burn in. Every 20th remaining scan was then used to construct a posterior mean estimate of \(C\). This estimate of \(C\) was essentially
identical to the one obtained using the s-likelihood, with an absolute difference between the two 
estimates of 0.0015, averaged across the 45 correlation parameters. This is quite small compared to 
the magnitude of the estimates, as can be seen in Figure 3, which plots the two estimates against 
one another.

Although the two estimation procedures produce essentially the same results for this large 
sample size of \( n = 2832 \), the existence and nature of potential differences for small sample sizes is 
not immediately clear. On one hand, the s-likelihood does not use all the information in the data, 
and so might result in less precise inference. On the other hand, using the full likelihood requires 
estimation of the univariate marginal CDFs, which for the GSS data amount to 164 additional 
parameters to estimate. Uncertainty in these unknown parameters could result in less precise 
inference for \( \mathbf{C} \). To explore these possibilities we undertook a simulation study to examine the 
sampling properties of the two different estimation methods. For each value of \( n_s \in \{25, 50, 100\} \) 
we generated 50 different GSS datasets of sample size \( n_s \) via simple random sampling of cases from 
the original set of \( n = 2832 \). An estimate of \( \mathbf{C} \) was then obtained for each of these \( 50 \times 3 \) 
datasets using both the s-likelihood and full likelihood approaches described above. We then examined the 
sampling bias and variance of these two estimators under the three different sample sizes, using the 
\( n = 2832 \) estimate as the truth.

The bias and sampling variance of the s-likelihood estimates are shown in Figure 4. The three 
horizontal line segments for each pair of variables represents the sampling mean plus and minus 
two sampling standard deviations of the estimator of the corresponding correlation for the three 
different sample sizes. Below these lines is a black square representing the estimate of the correlation 
using the full dataset. As the figure shows, the estimates are generally biased towards zero (the 
prior mean value), with bias decreasing as the sample size increase. Across the 45 correlation 
parameters, the mean squared error decreased by an average amount of 39\% in going from \( n_s = 25 \) 
to \( n_s = 50 \), and by an average amount of 43\% in going from \( n_s = 50 \) to \( n_s = 100 \).

The differences in MSE between the two approaches are described in Figure 5. The first panel 
compares the MSE in estimating the 45 correlation coefficients for \( n_s = 25 \). For most parameters 
(37 out of 45) the MSE is lower using the s-likelihood as opposed to the full likelihood. For \( n_s = 50 \) 
the MSE from the s-likelihood estimator is lower for 30 out of 45 parameters, but only 12 out of 
45 in the case of \( n_s = 100 \). One potential explanation of these results is that, for low sample sizes, 
the limited amount of information makes it difficult to accurately estimate the univariate marginal 
distributions, resulting in parameter estimates with high MSE when using the full likelihood. For 
moderate to large sample sizes, estimation of \( \mathbf{C} \) might be slightly improved by simultaneously 
estimating \( F_1, \ldots, F_p \), but for very large sample sizes such as \( n = 2832 \) the differences between the 
two procedures appear to be negligible.
5 Notions of sufficiency

The marginal set likelihood described above can be viewed as a generalization of marginal likelihood, a standard technique for dealing with nuisance parameters (see Section 8.3 of Severini (2000) for a review). One benefit of using such a likelihood is a gain in robustness, as inference no longer depends on assumptions about the relationship of the data to the nuisance parameters. Another benefit is a general simplification of the estimation problem, as the need to estimate a potentially high-dimensional set of parameters is eliminated. These benefits come at the cost of potentially losing information about the parameters of interest by only using part of the available data. Ideally, the statistic that generates the marginal likelihood is “partially sufficient” in the sense that it contains all relevant information in the data about the parameter of interest. Various definitions of partial sufficiency have been developed: Fraser (1956) defined $S$-sufficiency via properties of the marginal and conditional distributions of the statistic and the data. The concept of $G$-sufficiency was introduced in Barnard (1963) as a general principle for making inference about a parameter of interest when the inference problem remains invariant under a group of transformations. Rémond (1984) developed a generalization of these notions based on profile likelihoods called $L$-sufficiency, which has been refined and studied by Barndorff-Nielsen (1988, 1999). The general recommendation of these authors is to base inference for a parameter of interest on the sampling distribution of a statistic that is sufficient in some sense.

If $F_1, \ldots, F_p$ are all continuous then there are no ties among the data, and knowledge of $Z \in D$
Figure 4: Bias and variance of the Bayes estimates for $n \in \{25, 50, 100\}$ using the s-likelihood.
Figure 5: MSE comparison of the two methods for three different sample sizes.

provides a complete ordering of \( \{y_{1,j}, \ldots, y_{n,j}\} \) for each \( j \). This information is equivalent to the information contained in the ranks, and so \( \Pr(Z \in D|C) \) is equivalent to the sampling distribution of the multivariate ranks. Following the notation of [Rémont (1984)] we now show that the ranks \( r(Y) \) are a \( G \)-sufficient statistic in the sense of [Barnard (1963)]. Let \( C \in \mathcal{C} \) be the copula and \( F = \{F_1, \ldots, F_p\} \in \mathcal{F} \) the marginal distributions, and so the parameter space is \( \Omega = \mathcal{C} \times \mathcal{F} \) and the model space is \( \mathcal{P} = \{\Pr(\cdot|\omega) : \omega \in \Omega\} \), where \( \Pr(\cdot|\omega) \) is a probability measure on \( \mathbb{R}^p \) for each \( \omega \in \Omega \). Furthermore, let \( \mathcal{G} \) be the group of collections of \( p \) continuous strictly increasing functions, so that \( \mathcal{G} = \{G = (G_1, \ldots, G_p) : G_j \text{ is a continuous and strictly increasing function on } \mathbb{R}\} \). To each \( G \in \mathcal{G} \) there corresponds a one-to-one function on \( \mathcal{P} \) mapping \( \Pr(\cdot|\omega) \) to \( \Pr(G^{-1}(\cdot)|\omega) \) and the model space is closed under the action of \( \mathcal{G} \). As a result, \( \mathcal{G} \) induces a group \( \tilde{\mathcal{G}} = \{f_G : G \in \mathcal{G}\} \) on \( \Omega \) defined by \( \Pr(\cdot|f_G \omega) = \Pr(G^{-1}(\cdot)|\omega) \).

If the marginals are continuous the orbits of \( \Omega \) under \( \tilde{\mathcal{G}} \) can be put into 1-1 correspondence with \( \mathcal{C} \), and \( \mathcal{C} \) is therefore a maximal invariant parameter. Barnard defined a statistic \( t(Y) \) to be \( G \)-sufficient if it can be put into 1-1 correspondence with the orbits of \( \mathbb{R}^p \) under \( \tilde{\mathcal{G}} \). This is the case for the ranks \( r(Y) \) of \( Y \), and so \( r(Y) \) is said to be \( G \)-sufficient for estimation of \( C \). For continuous data, the marginal distribution of the ranks is equal to the partial set likelihood, and so basing inference on this likelihood function can been seen as using all available, relevant information in the \( G \)-sufficient sense.

A notion of sufficiency that is more directly related to maximum likelihood estimation is \( L \)-sufficiency: In the context of copula modeling, a statistic \( t(Y) \) is said to be \( L \)-sufficient for \( C \) if

A1. \( t(Y_0) = t(Y_1) \Rightarrow \sup_{\{F_1, \ldots, F_p\} \in \mathcal{F}} p(Y_0|C, F_1, \ldots, F_p) = \sup_{\{F_1, \ldots, F_p\} \in \mathcal{F}} p(Y_1|C, F_1, \ldots, F_p); \)

A2. \( p(t(Y)|C, F_1, \ldots, F_p) = p(t(Y)|C). \)
Note that the maximum likelihood estimate of \( C \) and its distribution will be a function only of an \( L \)-sufficient statistic, if one exists. If \( \mathcal{F} \) contains only continuous marginals, then one can show directly that the ranks \( r(Y) \) satisfy A1 and A2 (alternatively, \cite{Remon1984} shows that a \( G \)-sufficient statistic is also \( L \)-sufficient). Thus in the continuous case, the ranks are \( G \) - and \( L \)-sufficient, the MLE of \( C \) is a function of the ranks alone, and inference for \( C \) can be based on the distribution of the multivariate ranks, or equivalently, the s-likelihood.

If the marginals are allowed to be discontinuous then the orbits of \( \Omega \) under \( \mathcal{G} \) cannot be put into 1-1 correspondence with \( C \) and so \( C \) is not a maximal invariant. The problem is basically that if \( F_j(\cdot) \) is a discrete CDF, then \( F_j[G_j^{-1}(\cdot)] \) does not range over the space of all CDF’s as \( \mathcal{G} \) ranges over \( \mathcal{G} \). The ranks are no longer \( L \)-sufficient either: Condition A1 holds but A2 is violated because in the discrete case the distribution of the ranks depends on the marginal distributions. This means that estimation based on \( \Pr(r(Y)|C, F_1, \ldots, F_p) \) requires estimation of the nuisance parameters \( F_1, \ldots, F_p \). This may not be much of an issue if the number of levels of each variable is low, but for moderate numbers of levels we may wonder about the variability of the estimates due to the large number of parameters, or the need to specify a prior distribution for the marginals \( F_1, \ldots, F_p \) in the context of Bayesian estimation. In contrast, the s-likelihood based on \( \Pr(Z \in D|C) \) does not depend on \( F_1, \ldots, F_p \), thereby reducing the number of parameters to estimate and eliminating any need for a prior distribution on \( F_1, \ldots, F_p \). Furthermore, the s-likelihood is “sufficient” for continuous data but can be used with mixed continuous and discrete data. However, the concern remains that the s-likelihood may not be making full use of the information in discrete data about the copula parameters of interest. Although the asymptotic experiments in Section 4.2 may have alleviated this concern somewhat, it would still be desirable to describe precisely any potential information loss that results from using the s-likelihood as opposed to a full likelihood approach. Such a description could be obtained by comparing the curvatures of the s-likelihood and full likelihood surfaces, although the complicated parameter space and likelihood functions make description difficult except for the simplest of cases. A general description of the information properties of the s-likelihood in the context of copula estimation is a current research interest of the author.

6 Discussion

This article has presented an inferential procedure for copula parameters that can be applied to mixed continuous and discrete data. The procedure is based on a type of marginal likelihood, called an s-likelihood, which does not depend on the univariate marginal distributions of the data. The procedure therefore allows for the estimation of dependence parameters without the burden of having to estimate the marginal distributions. In an example, for small sample sizes the s-likelihood approach was seen to give parameter estimates having a lower MSE than those of a full likelihood
approach with nonparametrically estimated marginals. Differences between the two approaches were negligible when a large sample size was used.

Although this article has focused on semiparametric estimation of a Gaussian copula, the notion of s-likelihood is equally applicable to other copula models: Letting \( p(u|\theta) : \theta \in \Theta \) denote a parametric family of copula densities and \( \{y_{i,j} = G_j(u_{i,j}), i = 1 \ldots n, j = 1, \ldots, p\} \) be the observed data, the s-likelihood for \( \theta \) is given by \( \Pr(\max\{u_{k,j} : y_{k,j} < y_{i,j}\} < u_{i,j} < \min\{u_{k,j} : y_{i,j} < y_{k,j}\}, i = 1, \ldots, n, j = 1, \ldots, p|\theta) \). Given a prior distribution on \( \theta \), posterior inference can be obtained via a Markov chain Monte Carlo algorithm which iteratively resamples values of \( \theta \) and the \( u_{i,j} \)'s. However, full conditional distributions for these unknown quantities are generally hard to come by, and an MCMC sampler based on the Metropolis-Hastings algorithm is required for most models.

Code to implement the estimation strategy outlined in Section 3, written in the R statistical computing environment, is provided in the Appendix. A more detailed open-source software package is downloadable from R-archive at the following website:

http://cran.r-project.org/src/contrib/Descriptions/msgcop.html

### A R-code for Gaussian copula estimation

```r
# See also http://cran.r-project.org/src/contrib/Descriptions/msgcop.html
#
# Preconditions: Y, an n-observations by p-variables matrix
# S0, a p x p prior covariance matrix
# n0, an integer hyperparameter
# NSCAN, an integer number of iterations

########## helper functions
ldmvnorm<-function(Y,S) {  # log-density of a normal matrix
  n<-dim(Y)[1]
  p<-dim(Y)[2]
  -.5*n*log(det(S)) -.5*n*p*log(2*pi)-.5*sum( diag( solve(S)%*%t(Y)%*%Y))
}

rwish<-function(S0,nu){  # sample from a Wishart distribution
  sS0<-chol(S0)
  Z<-matrix(rnorm(nu*dim(S0)[1]),nu,dim(S0)[1])%*%sS0
  t(Z)%*%Z
}

########## starting values
n<-dim(Y)[1]
p<-dim(Y)[2]
set.seed(1)
```

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Z <- qnorm(apply(Y, 2, rank, ties.method = "random")/(n+1))
Zfill <- matrix(rnorm(n*p), n, p)
Z[is.na(Y)] <- Zfill[is.na(Y)]
Z <- t( (t(Z) - apply(Z, 2, mean))/apply(Z, 2, sd) )
S <- cov(Z)

########## constraints
R <- NULL
for(j in 1:p) { R <- cbind(R, match(Y[,j], sort(unique(Y[,j])))) }

########## start of Gibbs sampling scheme
for(nscan in 1:NSCAN) {

#### update Z[,j]
for(j in sample(1:p)) {
    Sjc <- S[j,-j]%*%solve(S[-j,-j])
    sdj <- sqrt( S[j,j] - S[j,-j]%*%solve(S[-j,-j])%*%S[-j,j] )
    muj <- Z[,-j]%*%t(Sjc)
    for(r in sort(unique(R[,j]))){
        ir <- (1:n)[R[,j]==r & !is.na(R[,j])]
        lb <- suppressWarnings(max( Z[R[,j]<r,j], na.rm=T))
        ub <- suppressWarnings(min( Z[R[,j]>r,j], na.rm=T))
        Z[ir,j] <- qnorm(runif(length(ir),
            pnorm(lb,muj[ir],sdj), pnorm(ub,muj[ir],sdj)), muj[ir], sdj)
    }
    ir <- (1:n)[is.na(R[,j])]
    Z[ir,j] <- rnorm(length(ir), muj[ir], sdj)
}

#### update S
S <- solve(rwish(solve(S0*n0+t(Z)%*%Z), n0+n))

########## end of Gibbs sampling scheme

References


