Effects of Collinearity

Collinearity can be interpreted theoretically and empirically.

Theoretically:
If \( E[y|x_1, x_2] \) linear in \( x_1, x_2 \), is \( E(y|x) \) linear in \( x \)?
If so, how are the linear relationships related?

Empirically:
How does including or excluding a variable change the cost for another?

Theoretical collinearity:
Suppose \( E[y|x_1, x_2] = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \)
\[ y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon \]
\( E(\epsilon|x_1, x_2) = 0 \)
Suppose we fit \( y \sim x_1 \) i.e. \( y = \beta_0 + \beta_1 x_1 + \epsilon \). Is this incorrect?

Case 0: \( x_2 = 0 \), then \( E[y|x_1, x_2] = E[y|x_1] = \beta_0 + \beta_1 x_1 \): correct

Case 1: \( x_2 \neq 0 \), \( x_2 \) is a deterministic experimental condition; incorrect

Case 2: \( x_2 \) is a randomly sampled quantity or random experimental condition; possibly correct

Suppose \( E[x_2|x_1] = \gamma_0 + \gamma_1 x_1 \)

Example:
\( x_1 \): Mean temperature
\( x_2 \): Max temperature

[Graph of linear relationship: mean vs. max temp]
In this case,

\[ Y = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \epsilon \]

\[
E[y|x_1] = \alpha_0 + \alpha_1 x_1 + E[\epsilon|x_1] = E[\epsilon|x_1]
\]

\[
= \alpha_0 + \alpha_1 x_1 + \alpha_2 \{ \gamma_0 + \gamma_1 x_1 \} + \epsilon
\]

\[
= (\alpha_0 + \alpha_2 \gamma_0) + (\alpha_1 + \alpha_2 \gamma_1) x_1
\]

\[
= \beta_0 + \beta_1 x_1. \quad \checkmark \quad \text{(new model is correct.)}
\]

Weather Example Revisited:

- \( x_1 \): Mean Temp
- \( x_2 \): Max Temp

Assume true values \( \alpha_1 \) small, neg., \( \alpha_2 \) large positive

Then \( \beta_1 = \alpha_1 + \alpha_2 \gamma_1 \) large positive

- True marginal effect of \( x_1 \) is large positive
- True conditional effect of \( x_1 \) is small negative

Marginal: Model is "marginalized" over possible values of \( x_2 \)

\[ E[y|x_1] = \int E[y|x_1, x_2] p(x_2|x_1) \, dx_2 \]

Conditional: Model is conditional on each value of \( x_2 \)

In this case, *both models are correct!*

\# Their interpretations are different

\# "effects" of a variable depend on the context.
Thought question:

Suppose \( E[y|x_1, x_2] = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 \)

\( E[x_2|x_1] = \delta_0 + \delta_1 x_1 \)

\( \therefore E[y|x_1] = (\alpha_0 + \alpha_2 \delta_0) + (\alpha_1 + \alpha_2 \delta_1) x_1 \)

- Conditional effect of \( x_1 \) is \( \alpha_1 \)

What is range of marginal effect \( \beta_1 \) as \( x_1 \) ranges from \(-\infty\) to \(\infty\)?

What is relationship between \( \alpha_1, \beta_1 \) when \( x_1 = 0 \)?

**Empirical Collinearity:** given data \( y, x_1, x_2 \)

Fit \( y \sim x_1 + x_2 \Rightarrow \hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 \)

\( y \sim x_1 \Rightarrow \hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 \)

**Questions:**

What is relation between \( \hat{\beta}_1, \hat{\beta}_2 \)?

How does \( x_2 \) affect \( \hat{\beta}_1, \hat{\beta}_2 \)?

**OLS Estimators:** for convenience, assume centered \( x_1, x_2 \)

\[ \hat{\beta}_1 = \frac{S_{xy}}{S_{x_1 x_2}} = \frac{\Sigma (x_i - \bar{x})(y_i - \bar{y})}{\Sigma (x_i - \bar{x})^2} = \frac{\Sigma x_i y_i}{\Sigma x_i^2} \]

\[ = \frac{\Sigma x_i y_i}{\Sigma x_i^2} = \frac{x_i \cdot y}{x_i \cdot x_i} \]
MLR: \( \hat{x} = (X^TX)^{-1}X^T \hat{y} \) \quad \hat{x} = \left( \begin{array}{c} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \end{array} \right)

\( X^T \hat{y} = \left( \begin{array}{c} 1 \\ x_{1,1}^T x_1 \\ x_{2,1}^T x_2 \end{array} \right) \left( \begin{array}{c} \hat{y} \end{array} \right) \left( \begin{array}{c} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \end{array} \right) \)

\( X^TX = \left( \begin{array}{ccc} 1 & x_{1,1}^T x_1 & x_{2,1}^T x_2 \\ x_{1,1}^T x_1 & 1 & 0 \\ x_{2,1}^T x_2 & 0 & 1 \end{array} \right) \left( \begin{array}{c} n \ 0 \ 0 \\ 0 \ x_{1,1}^T x_1 \ 0 \\ 0 \ 0 \ x_{2,1}^T x_2 \end{array} \right) \)

\( (X^TX)^{-1} \)

\( (X^TX)^{-1} X^T \hat{y} = \left( \begin{array}{c} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \end{array} \right) \hat{x}_0 = \frac{1}{n} \hat{y} = \bar{y} \)

\( \hat{x}_1 = \frac{1}{1x_1^T x_1 + |x_2^T x_2|^2} \left( x_2^T x_2 x_1 \bar{y} - x_1 x_2^T x_2 \right) \)

\( = \frac{x_1^T \hat{y} x_1}{x_1^T x_1} \left( \frac{x_2^T x_2}{(x_1^T x_1)(x_2^T x_2) - (x_1^T x_2)^2} - \frac{x_2^T \hat{y}}{x_2^T x_2} \right) \)

\( = \hat{\beta}_1 x_1 + \hat{\beta}_2 w_2 \)  \\

\( \hat{\beta}_2 = \frac{x_1^T \hat{y}}{x_2^T x_2} \)

Slope when only \( x_2 \) is in model

What happens when:
1. \( x_1^T x_2 = 0 \)
2. \( x_1^T x_2 > 0 \)
3. \( x_1^T x_2 < 0 \)

Example: Consider an... example.