You may use the following results without derivation:

- If $\theta \sim \text{beta}(a, b)$ then $E[\theta^c(1 - \theta)^d] = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a + c)\Gamma(b + d)}{\Gamma(a + b + c + d)}$.
- A $\chi^2_n$ distribution is a gamma($n/2, 1/2$) distribution.
- If $X \sim \text{gamma}(a, b)$ then $E[X^r] = \frac{\Gamma(a + r)}{\Gamma(a)} \frac{1}{b^r}$.

1. Let $X_1, \ldots, X_n \sim \text{i.i.d. } P_\theta \in \mathcal{P} = \{\text{uniform}(\theta - 1/2, \theta + 1/2), \theta \in \mathbb{R}\}$. Consider estimation of $\theta$ under squared error loss.
   
   (a) Show that this estimation problem is invariant under the group $\mathcal{G} = \{g : \mathbf{x} \rightarrow \mathbf{x} + a\mathbf{1}, a \in \mathbb{R}\}$, and find the induced groups on the parameter space and the decision space.

   (b) Find the UMREE of $\theta$, stating any results from the notes or text you are using.

2. (a) Let $X \sim \text{binomial}(n, \theta)$. Find the Bayes estimator of $\theta$ under the uniform prior and the loss $L(\theta, d) = (d - \theta)^2/[\theta(1 - \theta)]$. Find the non-Bayes risk function of this Bayes estimator under this loss.

   (b) Show that $\bar{X} = X/n$ is the minimax estimator of $\theta$ under this loss. State any theorems you use to obtain the result.

   (c) Suppose $\delta$ is minimax for a quantity $\theta = \theta(P)$ for $P \in \mathcal{P}_0$. Find a simple condition on $\delta$ that makes $\delta$ minimax for a larger family $\mathcal{P}_1$ where $\mathcal{P}_0 \subset \mathcal{P}_1$, and prove your result.

   (d) Let $X_k \sim \text{binomial}(n, \theta_k)$, $k = 1, \ldots, K$, with $X_1, \ldots, X_K$ being independent. Find the minimax estimator of $\bar{\theta} = \sum \theta_k/K$ under the loss $L(\bar{\theta}, d) = (d - \bar{\theta})^2/[\bar{\theta}(1 - \bar{\theta})]$. State any theorems you use to obtain the result.

3. Consider estimation of $\theta$ from $\mathbf{X} \sim N_p(\theta, \mathbf{I})$ under sum-of-squared errors loss.
(a) Under the prior $\theta \sim N_p(0, \tau^2 I)$, obtain the Bayes estimator $\delta_\pi$ and its Bayes risk $R(\pi, \delta_\pi)$.

(b) Show that the posterior risk of any estimator $\delta$ of $\theta$ can be expressed in terms of the posterior variance of $\theta$ and the posterior bias of $\delta$ (the difference between $\delta$ and $E[\theta|X]$).

(c) Obtain the marginal (prior predictive) distribution of $X$, and from this and (b) calculate the Bayes risk of $\delta_{JS} = (1 - (p-2)/(X \cdot X))X$.

(d) Obtain a lower and upper bound on $R(\pi, \delta_{JS})$ based on $R(\pi, \delta_\pi)$ that is valid for all values of $\tau^2$. Discuss the implications of this result.

4. Suppose an unknown parameter $\theta$ is either $1/2$ or $1/3$. Our goal is to estimate $\theta$ with zero-one loss using the information from a single binary($\theta$) random variable $X$. Consider the following four non-randomized estimators:

$\delta_1(X) = 1/3$
$\delta_2(X) = 1/(3 - X)$
$\delta_3(X) = 1/2$
$\delta_4(X) = 1/(2 + X)$

(a) Find the risk functions of each estimator (there are only two possible values of $\theta$).

(b) For each estimator, determine whether or not it is
   i. admissible among non-randomized estimators;
   ii. minimax among non-randomized estimators.

(c) Calculate the Bayes risk of each estimator as a function of the prior $\pi = \Pr(\theta = 1/2)$. For what values of $\pi$ is each estimator Bayes?

(d) Note that a randomized estimator for this problem can be expressed as

$$\delta(X) = \begin{cases} 
1/2 & \text{w.p } a(X) \\
1/3 & \text{w.p } 1 - a(X).
\end{cases}$$

Calculate the risk function of such estimators as a function of $\theta$, $a_0 = a(0)$ and $a_1 = a(1)$, and determine if one of $\delta_1, \ldots, \delta_4$ is still minimax among all estimators (randomized and non-randomized).