Stat 581 Midterm Exam Solutions (Autumn 2012)

1. See the course notes.

2. (a) We have squared error loss, so

\[
R(\theta, aY) = \text{Var}(aY) + \text{Bias}(aY)^2 \\
= a^2 v(\theta) + (1 - a)^2 \theta^2.
\]

For \(v(\theta) = \theta^2\), the risk is \(a^2 \theta^2 + (1 - a)^2 \theta^2\), which is minimized by \(a = 1/2\). Thus \(\delta^*\) has minimum risk in \(\mathcal{A}\) and so is admissible within the class. Also, we have

\[
R(\theta, Y/2) = \theta^2/2 < \theta^2 = R(\theta, Y),
\]

so \(\delta^*\) dominates \(\delta_1\).

(b) For \(v(\theta) = \theta\), the risk is \(a^2 \theta + (1 - a)^2 \theta^2\), which is minimized by \(a = \theta/(1 + \theta)\). Thus \(aY\) is uniquely minimizes risk when \(\theta = a/(1 - a)\), and so no member of \(\mathcal{A}\) dominates another. Furthermore, a few calculations show that

\[
R(\theta, aY) \leq R(\theta, Y) \quad \text{if} \quad 0 < \theta \leq (1 + a)/(1 - a), \quad \text{but} \\
R(\theta, aY) > R(\theta, Y) \quad \text{if} \quad (1 + a)/(1 - a) < \theta
\]

so no member of \(\mathcal{A}\) dominates \(\delta_1\) for this variance function.

(c) The Bayes risk of any estimator in \(\mathcal{A}\) is

\[
R(\pi, aY) = \mathbb{E}[R(\theta, aY)] \\
= \mathbb{E}[a^2 \theta^k + (1 - a)^2 \theta^2] \\
= a^2 k! + 2(1 - a)^2
\]

This is minimized in \(a\) by \(a = 2/(2 + k!)\), so the Bayes estimator is \(2Y/(2 + k!)\).

3. (a) The admissible estimators are on the lower left-hand boundary of the set, i.e. those points \(s \in \mathbb{R}^2\) for which \(S \cap \{x \in \mathbb{R}^2 : x_1 \leq s_1, x_2 \leq s_2\} = s\).

(b) For this you needed to plot a vector in the positive quadrant of \(\mathbb{R}^2\).
(c) and (d) It might be helpful to do (d) first. Extend out the vector $\pi$ by some multiple $r$ at least until the line perpendicular to it has an intersection with $\mathcal{S}$. Draw the perpendicular line. The intersection of this line with $\mathcal{S}$ gives you part (d). To get (c), imagine sliding your perpendicular down in the direction of $-\pi$, until it has only a single point of intersection with $\mathcal{S}$. This point is your Bayes estimator.