Lecture Notes 1

Statistical Functionals

The Gâteux derivative of a statistical functional \( T(F) \) is the limit
\[
d_1 T(F;G - F) = \lim_{\lambda \downarrow 0} \frac{T(F + \lambda(G - F)) - T(F)}{\lambda}.
\]
If \( Q(\lambda) = T(F + \lambda(G - F)) \) has a McLaurin expansion, we get an expansion (the von Mises expansion) of \( T \) by noting that
\[
Q(0) = T(F), \quad Q(1) = T(G), \quad Q'(0) = d_1 T(F;G - F)
\]
etc., yielding
\[
T(G) = T(F) + \sum_{k=1}^m \frac{1}{k!} d_k T(F;G - F) + R_m(G).
\]
We are usually particularly interested in \( G = F_n \), and we write \( R_m(F_n) = R_{m,n} \).

\( T \) has a differential at \( F \) with respect to a norm \( \| \cdot \| \) if there is a linear functional \( \hat{T}(F;\Delta) \) such that for all \( G \)
\[
T(G) - T(F) - \hat{T}(F;G - F) = o(\|G - F\|).
\]
\( \hat{T} \) is called the Frechét derivative of \( T \).

**Theorem 1:** If \( T \) has a differential at \( F \) with respect to \( \| \cdot \| \), then for any \( G \) the Gâteux derivative \( d_1 T(F;G - F) \) exists and equals \( \hat{T}(F;G - F) \).

**Theorem 2:** Let \( T \) have a differential at \( F \) with respect to \( \| \cdot \| \). Let \( X_1, \ldots, X_n \) be observations from \( F \) (not necessarily independent), such that \( \sqrt{n} \| F_n - F \| = O_p(1) \). Then \( \sqrt{n} R_{m,n} \to 0 \).

Define the influence curve of \( T \) at \( F \) by \( h(F; x) = d_1(F, F_x - F) \), where \( F_x \) is the cdf of point mass at \( x \).

**Theorem 3:** Suppose \( T \) has a linear derivative satisfying
(a) \( 0 < \text{Var}_F h(F;X) < \infty \)
(b) \( \sqrt{n} R_{m,n} \to 0 \).

Define \( \mu(T,F) = E_F h(F;X) \) and \( \sigma^2(T,F) = \text{Var}_F h(F;X) \). Then
\[
T(F_n) \sim \text{AsN}(T(F) + \mu(T,F), \sigma^2(T,F)/n).
\]
Theorem 4: Assume that $T$ has an influence curve which is identically zero, and a bilinear second Gâteaux derivative with symmetric kernel $h(F;u,v)$ such that

(a) $0 < \text{Var}_P h(F;X_1,X_2) < \infty$

(b) $nR_{2n} \to 0$

(c) $E_{F}(F;X,X) = 0$ as a function of $x$.

Then

$$n(T(F_n) - T(F)) \xrightarrow{d} \frac{1}{2} \sum_{j=1}^{\infty} \lambda_j V_j$$

where the $V_j$ are iid $\chi_1^2$-random variables, and $\lambda_j$ is the eigenvalue of the operator

$$A[\hat{g}](x) = \int (h(F,x,y)g(y)dF(y)$$

corresponding to the eigenfunction $g_j(x)$. 