1. Let $T$ be a location M-estimator, and define $\lambda(t,F) = \int \psi(u-t) dF(u)$.

(a) Show that if $\psi$ is bounded and $\lambda(t,F_0)$ has a unique zero at $t = T(F_0)$ then $T(F) \to T(F_0)$ as $d_L(F,F_0) \to 0$.

*Hint:* Prove that $\lambda(t+\varepsilon,F_0) - \|\psi\| \varepsilon \leq \lambda(t,F) \leq \lambda(t-\varepsilon) + \|\psi\| \varepsilon$ where $\|\psi\| = \psi(\infty) - \psi(-\infty)$.

(b) Show that if $\psi$ is not bounded then $T(F)$ need not converge to $T(F_0)$.

(c) Show that if $T(F_0)$ is not unique then $T(F)$ need not converge to $T(F_0)$.

2. Let $T(F) = \int F^{-1}(t) dM(t)$ where $M$ is a cdf on $[\alpha,1-\alpha]$, $0 < \alpha \leq \frac{1}{2}$, and $\alpha$ is taken as large as possible, and $F$ is a cdf on the reals with positive density.

(a) Describe $T(F_n)$.

(b) Show that the influence curve for $T$ is

$$IC(x) = \int \frac{sdM(s)}{f(F^{-1}(s))} - \int \frac{dM(s)}{f(F(x))}.$$

(c) Show that the breakdown point of $T$ is $\alpha$.

*Hint:* The worst case is the same as for location M-estimators.

3. Let $T$ be an M-estimator (not necessarily of location), so $\int \psi(x;T(F)) dF(x) = 0$

Assuming that $n^{1/2} (T(F_n) - T(F)) \to_d N(0, \sigma_F^2)$, evaluate $\sigma_F^2$ and show that $1/\sigma_F^2$ is a convex function on $F$.

4. Suppose we have iid observations of a positive random variable with distribution $F(x) = (1-\varepsilon)F_0(x/\hat{s}_\alpha) + \varepsilon H(x)$ where $F_0$ is the nominal parametric model with unknown scale parameter $s_0$ and $H$ is an arbitrary contamination distribution on the positive real line. Define an M-estimator of scale as $\hat{s}_n = S(\chi,F_n)$ where

$$S(\chi,F) = \inf \left\{ s : \int_0^\infty \chi(x/s) dF(x) \leq b \right\}$$

and $\chi$ is a non-decreasing function.

(a) Show that $b = E_{F_0(\chi(X))}$ implies Fisher consistency at $F_0$.

(b) Show that $\hat{s}_n$ is scale equivariant.

(c) Let $\chi = 1(X > a)$. Determine $b$ and $\hat{s}_n$. 