Solution 753

1. \( X \) and \( Y \) are not independent, since if I know \( X \) I can figure out \( U \) and therefore \( Y \).

\[
E(2XZ) = E(XY) = \int_0^\infty \sin(2\pi u) (u)(2\pi u) \, du
\]

\[
= \int_0^\infty \frac{1}{2} \sin(4\pi u) \, du = \frac{1}{2} \left[ \cos(4\pi u) \right]_0^\infty = 0
\]

\( E(X) = \int_0^\infty \sin(2\pi u) \, du = 0 = E(Y) \) so \( \cos(x, y) = 0 \) \( \rightarrow \) \( 0 = 0 \)

2. Error \( Y = \sum X_i \) where \( X_i \sim U(-5, 5) \)

so \( E(X_i) = 0 \) and \( \text{Var}(X_i) = \frac{1}{12} \)

\( \text{Var}(X) = \frac{100}{12} \)

\( P(1 / X > 5) \leq \frac{100}{6 / 12.5^2} = \frac{1}{3} \) assuming independence of roundoff. May not be 5-year.

3. Total resistance \( Y = \sum X_i \) \( \text{Var}(Y) = 6 \text{Var}(X) \)

assuming that the resistors are independent.

So \( \sqrt{6} \sigma^2 \leq 0.4 \quad \sigma \leq \frac{0.4}{\sqrt{6}} = 0.16 \)

Assuming reasonable independence of resistor.

4. \( T = \sum X_i \) where \( X_i \sim \text{Exp}(1) \) with prob \( \frac{1}{1-p} \)

\( E(T) = \sum E(X_i) \quad \text{Var}(T) = \sum \text{Var}(X_i) = \sum i^2 p(1-p) = \frac{n(n+1)(2n+1)}{6} \beta(1-p) \)

A fair price for \( n \) games would be \( \frac{n(n+1)^\frac{1}{2}}{2} \).

5. \( P(\min X_i > t) = P(\text{all } X_i > t) = (e^{-\lambda t})^n = e^{-\lambda nt} \)

so \( F_{\min X_i}(t) = 1 - e^{-\lambda nt} \) \( \text{Var}(X_i) = \lambda^2 e^{-\lambda nt} \)

and \( E(\min X_i) = \frac{\lambda}{\lambda n} \)

6. The prices are \( X \) and \( 1 - X \).

\( E\left( \max (X, 1-x) - \min (X, 1-x) \right) = \int_0^1 \left( \max(u, 1-u) - \min(u, 1-u) \right) \, du \)

\( = \int_0^1 (2u - u - u) \, du = \int_0^1 (u - u - u) \, du \)

\( = \int_0^1 (u - u - u) \, du = \int_0^1 (u - u - u) \, du \)
\[
= \int_0^{1/2} (1-2u) \, du + \int_{1/2}^1 (2u-1) \, du = \left[ u - u^2 \right]_0^{1/2} + \left[ \frac{u^2 - u}{3/2} \right]_{1/2}^1
\]

\[
= \frac{1}{2} - \frac{1}{4} + 1 - \frac{1}{2} = \frac{1}{4} + \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}
\]

7. Assuming that bricks and mortar are independent, we have

\[
E(\text{length}) = 30 \times 10 + 49 \times \frac{1}{2}
\]

\[
\text{Var}(\text{length}) = \frac{50}{32^2} + \frac{49}{16^2} = 0.24 \quad \text{so} \quad \text{sd}(\text{length}) = 0.49
\]

8. \(E(aX+b) = aE(X+b)\) so

\[
E\left( (aX+b-aE(X)+b) \right) = a^3E((X-E(X))^3) \quad \text{while} \quad \text{sd}(aX+b) = |a| \cdot \text{sd}(X) \quad \text{(from class)}
\]

so the skewness coefficient is

\[
\frac{a^3 \cdot E((X-E(X))^3)}{|a|^3 \cdot \text{sd}(X)} = \text{sgn}(a) \cdot \text{skew}(X)
\]

The invariance is that it does not depend on \(b\) is because it is a function of the shape of the density, not the location of it.

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#6: Let \(X = \text{length of the shorter piece}\)

\(X \sim \text{Uniform}[0,1/2] \quad \text{Note:} \quad E(X) = \frac{1}{4}\)

Difference = Longer - Shorter = \(2x[\frac{1}{2} - X]\)

\[
E(\text{Difference}) = 2 \times \left[ \frac{1}{2} - E(X) \right] = 2 \times \left[ \frac{1}{2} - \frac{1}{4} \right] = \frac{1}{2}
\]