Solution Problem Set 4

1. \( P(Y \leq y) = P(F(X) \leq y) = P(X \leq F^{-1}(y)) = F_X(F^{-1}(y)) = y \) so \( Y \sim U(0,1) \)

2. \( 1 = E(S_n/S_n) = E(X_1/S_n + X_2/S_n + \ldots + X_n/S_n) = n \frac{E(X_i)}{S_n} \) so \( E(X_i/S_n) = \frac{1}{n} \) and \( E(S_n/S_n) = E\left( \frac{1}{n} \sum X_i/S_n \right) = mE(X_i/S_n) = \frac{m}{n} \)

3. (a) The uniform distribution on the square has \( P(\Omega \in A) = \text{area of } A \) so here

\[
\int_{f(x)}^1 P(\Omega \text{ below } f) = \int_0^1 f(x) \, dx
\]

(b) Draw \( n \) independent uniform rv's from the square. The relative frequency of hits converges in probability to the integral sought.

4. (a) \( P(\text{second black}) = P(\text{second black/first black}) P(\text{first black}) + P(\text{second black/first red}) P(\text{first red}) = \frac{b}{r+b+1} \cdot \frac{b}{r+b} + \frac{b}{r+b+1} \cdot \frac{r}{r+b} = \frac{b}{r+b+1} \cdot \frac{b+r}{r+b} \)

The same holds for any draw, when we do not know the outcome of any other draw.

(b) \( S_n = \frac{1}{n} \sum X_i \) \( E(S_n) = n E(X_i) = n \frac{b}{r+b} \)

(c) \( E(X_1X_2) = P(X_1 = X_2 = 1) = \frac{b}{r+b+1} \cdot \frac{1}{r+b+1} \)

so \( \text{Cov}(X_1, X_2) = \frac{b}{(r+b)^2} - \frac{b^2}{(r+b+1)^2} = \frac{rb}{(r+b)(r+b+1)} \)

\[
\text{Var}(X_1) = \frac{b}{r+b} - \left( \frac{b}{r+b} \right)^2 = \frac{rb}{(r+b)^2} = \text{Var}(X_2)
\]
& \text{Cov}(X_1, X_2) = \frac{rb}{(r+b)^2(r+b+1)} - \frac{rb}{(r+b)^2} = \frac{1}{(r+b+1)} \\
(d) \var(\sum_{i=1}^{n} x_i) = n \var(X_1) + n(n-1) \text{Cov}(X_1, X_2) \\
= \frac{n \cdot rb}{(r+b)^2} + \frac{n(n-1) \cdot rb}{(r+b)^2(r+b+1)} \\
= \frac{nr b}{(r+b)^2} \left(1 + \frac{n-1}{r+b+1}\right)