Lecture Notes 1

Statistical Functionals

The Gâteux derivative of a statistical functional $T(F)$ is the limit

$$d_1T(F;G-F) = \lim_{\lambda \downarrow 0} \frac{T(F+\lambda(G-F)) - T(F)}{\lambda}.$$

If $Q(\lambda) = T(F+\lambda(G-F))$ has a McLaurin expansion, we get an expansion (the **von Mises expansion**) of T by noting that $Q(0)=T(F)$, $Q(1)=T(G)$, $Q'(0+) = d_1T(F;G-F)$ etc., yielding

$$T(G) = T(F) + \sum_{k=1}^{m} \frac{1}{k!} d_kT(F;G-F) + R_m(G).$$

We are usually particularly interested in $G = F_n$, and we write $R_m(F_n)=R_{mn}$.

T has a **differential** at F with respect to a norm $\|\cdot\|$ if there is a linear functional $\dot{T}(F;\Delta)$ such that for all G

$$T(G) - T(F) - \dot{T}(F;G-F) = o(\|G-F\|).$$

$\dot{T}$ is called the Frechét derivative of T.

Theorem 1: If T has a differential at F with respect to $\|\cdot\|$, then for any G the Gâteux derivative $d_1T(F;G-F)$ exists and equals $\dot{T}(F;G-F)$.

Theorem 2: Let T have a differential at F with respect to $\|\cdot\|$. Let $X_1,\ldots,X_n$ be observations from F (not necessarily independent), such that $\sqrt{n}\|F_n - F\| = O_P(1)$. Then $\sqrt{n}R_{1n} \xrightarrow{P} 0$.

Define the **influence curve** of T at F by $h(F;x) = d_1(F,\varepsilon_x - F)$, where ε_x is the cdf of point mass at x.

Theorem 3: Suppose T has a linear derivative satisfying
(a) $0 < \mathrm{Var}_F h(F;X) < \infty$
(b) $\sqrt{n}R_{1n} \xrightarrow{P} 0.$
Define $\mu(T,F) = E_F h(F;X)$ and $\sigma^2(T,F) = \mathrm{Var}_F h(F;X)$. Then

$$T(F_n) \sim \mathrm{AsN}(T(F) + \mu(T,F), \sigma^2(T,F)/n).$$

Theorem 4: Assume that T has an influence curve which is identically zero, and a bilinear second Gâteux derivative with symmetric kernel $h(F;u,v)$ such that

(a) $0 < Var_F h(F;X_1,X_2) < \infty$

(b) $nR_{2n} \xrightarrow{P} 0$

(c) $E_F(F;x,X) = 0$ as a function of x.

Then

$$n(T(F_n) - T(F)) \xrightarrow{d} \tfrac{1}{2}\sum_{j=1}^{\infty} \lambda_j V_j$$

where the V_j are iid χ_1^2-random variables, and λ_j is the eigenvalue of the operator

$$A[g](x) = \int (h(F:x,y)g(y)dF(y)$$

corresponding to the eigenfunction $g_j(x)$.