ACCUMULATED CLAIMS AND COLLECTIVE RISK IN INSURANCE:
HIGHER ORDER ASYMPTOTIC APPROXIMATIONS

BY

KNUT K. AASE

TECHNICAL REPORT NO. 45
APRIL 1984

DEPARTMENT OF STATISTICS
UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195
ACCUMULATED CLAIMS AND COLLECTIVE RISK IN INSURANCE: HIGHER ORDER ASYMPTOTIC APPROXIMATIONS†

by

Knut K. Aase
Norwegian School of Economics
and Business Administration
Bergen, Norway

ABSTRACT

A stochastic process modelling the reserves of an insurance company is introduced. After a slight modification, the model can be reinterpreted for the analysis of the amount of claims against an insurance company as well. The flexibility this model provides may prove useful when approaching real problems in the insurance industry. With some simplifying assumptions, it becomes possible to compute the probability distribution of the claims at each time epoch. The expression for the exact distribution becomes complicated after a long period of time has elapsed. Hence the need for asymptotic expansions arises.

Conditions are found under which a higher order asymptotic expression is valid. The Berry-Esseen approach is used in the proof of this result. There are problems with establishing an expansion for densities. These are briefly discussed.

A result is provided for approximating the tails of the distribution \( F(x;t) \). The result constrains the size of \( |x| \), as it must be compared to \( t \).

Finally, an equation for the probability of ruin is derived. The model used here allows, among other things, for uncertainty in the premium income of the company. This aspect adds realism to the model, since the concept of uncertainty is essential when analyzing the market.

†Research was partially supported by National Science Foundation Grant No. MCS-83-02337.

April 24, 1984
1. INTRODUCTION

1.1 The main part of the present paper deals with computing higher order approximations for probability distributions occurring in insurance. Lundberg's macromodel implies that the accumulated claims \( X(t) \) against an insurance company at time epoch \( t \) is a compound Poisson process. This model is generalized in section 2, where we take advantage of newer developments in continuous-time stochastic modelling. It may prove useful as the need for more realistic models in the insurance industry grows.

1.2 Section 3 deals with higher order asymptotic expansions. The model is a simplified version of the one presented in section 2. This version is more complex than the Lundberg model but is still parsimonious. Specifically, it adds more realism when analyzing uncertainty in the insurance market. An Edgeworth expansion is established, approximating the claim distribution \( F(x; t) \) to the order \( o\left(t^{-\frac{1}{2}}\right) \), \( r \geq 3 \), uniformly in \( x \). The conditions used are weak and are likely to be valid when applied to insurance problems. We use the Berry-Esseen approach in which Essén's smoothing lemma plays a crucial role. Though Cramér's pioneer work in the area is likely to be better known to actuarians, his methods are now obsolete.

1.3 In section 4 we approximate the tails of the distribution \( F(x; t) \) i.e. approximations for \( |x| \) large. There Edgeworth expansions are abandoned, since they do not perform well in the tails. Instead, the result gives a range of \( x \)-values where the normal approximation may be used. An estimate of the error committed in using this approximation is also given. The range of \( x \)-values given by this result covers many situations of practical interest.

1.4 In section 5 we treat ruin problems. We use the same random model as in the previous sections. In this context an integral equation for the ruin probability is derived. We established some results in probability theory in order to find this equation.

1.5 To indicate the usefulness of higher order asymptotic expansions, consider the Lundberg model. The probability distribution function (p.d.f.) of \( X(t) \) is an infinite weighted sum of convolutions

\[
P[X(t) \leq x] = F(x; t) = e^{-\lambda t} \sum_{n \geq 0} \frac{(\lambda t)^n}{n!} F^n(x)
\]  

(1.1)

Here \( \lambda \) is the Poisson intensity of the claims. Given such a claim, \( F(x) \) is the conditional p.d.f. of the individual claim and \( F^n(x) \) is the \( n \)-fold convolution of \( F(x) \) with itself. If \( t \) is large, (1.1) may be very complicated to use in practice. Example: If \( \lambda = 1 \), \( t = 15 \), as many as 38 terms in (1.1) may be needed for an answer, correct to six decimal places. Here Edgeworth expansions can provide substantial computational simplifications.

April 24, 1984
addition, they usually perform excellently, except in the tails of the distribution. Example: Assume $F(x)$ is Chi-square with four degrees of freedom, $\lambda = \frac{1}{4}$ and $t = 2$. Then $F(7;2) = 0.90556$, correct to five decimal places by using seven terms in the series (1.1). An Edgeworth expansion to order $o(t^{-2})$ yields $F(7;2) = 0.89599$, a fairly good approximation, even for $t$ small.

These expansions offer the advantage that we need not know $F(x)$, only its first $r$ moments (cumulants). These are much easier to estimate from observed data than the conditional p.d.f. itself, as they require large collections of data in order to be estimable with a reasonable degree of accuracy. In the example above, only the first four cumulants of $F(x)$ were used.

1.6 The Lundberg model implies that

$$X(t) = x_0 + \mu t - V(t)$$

(1.2)

where $V(t)$ stands for the accumulated amount of claims in $[0,t]$ against the insurance company. The rate (sure rate) of income from premiums equals $\mu$, and $x_0$ is the security reserve at epoch 0. $X(t)$ thus represents the company's total reserve at time $t$, and ruin stands for a negative reserve.

Let $R(x_0)$ be the probability that no ruin ever occurs. Assuming $V(t)$ to be a compound Poisson process, the following equation may be derived:

$$R(x_0) = \frac{\lambda}{\mu} \int_{x_0}^{\infty} e^{-\lambda(s-x_0)} ds \int_{-\infty}^{s} R(s-x)F(dx)$$

(1.3)

Assuming $R(x_0)$ to be differentiable, this equation can be transformed to an integro-differential equation, which can be integrated to obtain a renewal equation (at least when $F(x)$ is concentrated on $[0,\infty)$). It is, however, easy to construct examples where this procedure cannot be valid: Suppose $F(x)$ has an atom at $x = 1$ of mass 1, and set $\lambda = 1$. The solution to (1.3) is given by

$$R(x_0) = \begin{cases} 
(1 - \frac{1}{\mu}) e^{x_0/\mu} & , 0 \leq x_0 \leq 1 \\
(1 - \frac{1}{\mu}) e^{x_0/\mu} - \frac{1}{\mu} (1 - \frac{1}{\mu}) (x_0 - 1) e^{x_0/\mu} & , 1 \leq x_0 \leq 2 \\
\text{etc.}
\end{cases}$$

(1.4)

Clearly different results for $R'(1)$ are obtained depending upon which one of the above expressions in (1.4) are being differentiated.

If $F(dx) \ll dx$, i.e. if $F(x)$ has a probability density, the renewal approach is valid. In this case the following asymptotic estimate holds, as $x_0 \to \infty$

$$R(x_0) \sim 1 - \frac{1}{kp}(1 - \frac{\lambda}{\mu}) e^{-x_0}$$

(1.5)
Here \( k \) is such that
\[
\frac{\lambda}{\mu} \int_0^\infty e^{kx}(1-F(x))dx = 1 \quad \text{and} \quad p = \int_0^\infty xF(dx), \quad p^* = \frac{\lambda}{\mu} \int_0^\infty e^{kx}x(1-F(x))dx < \infty.
\]

The famous estimate in (1.5) was first obtained by Cramér using a different approach.

1.7 In a subsequent paper we consider the problem of heterogeneity in non-life insurance. Specifically, we are concerned with the statistical estimation of the risk distribution of \( \lambda \), where \( \lambda \) is allowed to vary across the population of policies. An early reference here is Granander (1957). This approach may alternatively interpret \( \lambda \) as a latent variable, as in Andersen (1982). This kind of structure has been given some attention in recent economic literature. Various methods are considered, among them kernel-estimation techniques for densities.

April 24, 1984
2. THE STRUCTURE OF THE RANDOM MODEL

2.1 In this section we present a stochastic model that seems reasonable in various problems in the insurance industry. The author has presented related models in other economic applications where uncertainty plays a major role. They include portfolio optimization projects (Aase (1983a), (1983b), (1984a)) and risky R & D projects (Aase (1984b)).

2.2 Given is the filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P) \) and a stochastic integral equation

\[
X(t) = x_0 + \int_0^t \mu(s,X)ds + \int_0^t \sigma(s,X)dB + V(t)
\]

where

\[
V(t) = \int_0^t \int_R \sum_{k=1}^d \gamma_k(t,X,y) \nu_k(dt,dy)
\]

First we give two interpretations of these equations which could be of interest to insurance:

(a) \( X(t) \) represents the company's total reserve at time epoch \( t \). Here \( x_0, \mu \) and \( V(t) \) have the same meaning as in (1.2).

(b) \( X(t) \) stands for the total amount of claims against the company in \([0,t]\).

The basic difference from (a) being that \( V(t) \) has the opposite sign. Still \( V(t) \) represents the larger claims. However, the first three terms on the right-hand side of (2.1) represent the smaller claims, where each one individually does not matter much to the company.

The results in sections 3 and 4 can be applied to both (a) and (b).

In section 5 we have case (a) in mind.

2.3 As for the interpretations of (2.1) and (2.2), the three terms on the right of (2.1) give the dynamics of the sample continuous part of \( X \). \( B(t) \) is a standard Brownian motion (Wiener process). (2.2) describes the dynamics of the jump part of \( X \). \( \nu = (\nu_1, \nu_2, \ldots, \nu_d) \) is a \( d \)-dimensional random counting measure. \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d) \) represents the sizes of the different kinds of claims. More precisely, there exists an imbedded marked point process \( N_t(A) = (N_{1t}(A), N_{2t}(A), \ldots, N_{dt}(A)) \), with orthogonal components. \( A \subset R \). (See e.g. Bremaud (1981)). Let \( T_{k,n} \) be the time points of claims of type \( k \), where the size of the claim is \( I_{k,n} \), \( n \geq 1 \), \( k = 1, 2, \ldots, d \). Then (2.2) can be written as follows

\[
V(t) = \int_0^t \int_R \sum_{k=1}^d \gamma_k(s,X,y) \nu_k(ds,dy)
\]

\[
= \sum_{k=1}^d \sum_{n \geq 1} \gamma_k(T_{k,n},X,T_{k,n}) 1(T_{k,n} \leq t)
\]

April 24, 1984
where \( 1(E) \) is the indicator function of the event \( E \in \Xi \).

It is assumed that there exists a \( d \)-dimensional intensity kernel \( \lambda(t,X,y) \) such that
\[
E(v(t,A) \mid \Xi_t) = \int_0^t \int_A \lambda(s,X,dy)ds
\]  
(2.4)

Here \( d \) equals the number of different types or groups of policies. The corresponding intensity kernels are \( \lambda(t,X,y) = (\lambda_1, \lambda_2, \ldots, \lambda_d)(t,X,y) \), where
\[
\lambda_k(t,X,y) = \lambda_k(t)F_{k,t}(dy), \quad k = 1, 2, \ldots, d
\]  
(2.5)

\( \lambda_k(t) \) is the intensity of the marked point process \( N_{kt}(A) \). Given a claim of type \( k \) at time \( t \), \( F_{k,t}(dy) \) is the conditional individual claim p.d.f., i.e. \( F_{k,T_{k,n}}(A) = P[Y_{k,n} \in A \mid \Xi_{T_{k,n}}] \).

Note that the actual size of the claim of type \( k \) at time \( T_{k,n} \) is not \( Y_{k,n} \) but \( \gamma_k(T_{k,n},X,Y_{k,n}) \). Only in the special case where \( \gamma_k(t,X,y) = \pm y \) the size of the claim is \( Y_{k,n} \). In the interpretation (a) the minus sign is used and in case (b) the plus sign is utilized.

2.4 Here we introduce a sequence of \( d \)-valued random variables (r.v.'s) \( Z_n \), where \( Z_n = k \) if the \( n \)th claim is of type \( k \). Then
\[
n_k(t) = \sum_{n \geq 1} 1(T_n \leq t)1(Z_n = k)
\]  
(2.6)

equals the number of claims of the \( k \)th kind in \([0,t]\), \( T_n \) being the time points in which claims are made. An interesting interpretation follows from
\[
\frac{\lambda_k(T_n)}{\lambda(T_n)} = P[Z_n = k \mid \Xi_{T_n}]
\]  
(2.7)

where \( \lambda(t) = \sum_{k=1}^d \lambda_k(t) \) is the \( \Xi_t \)-intensity of \( n(t) = \sum_{k=1}^d n_k(t) \). Given \( \Xi_{T_n} \) and that there is a jump of one of the \( n_k(t) \)'s at time \( t \), \( \lambda_k(t)/\lambda(t) \) is the probability of having a jump of the counting process \( n_k(t) \) at time \( t \). With this in mind, one may see that
\[
F_t(dy) = \sum_{k=1}^d \lambda_k(t)F_{k,t}(dy)/\lambda(t)
\]  
(2.8)

is the conditional individual claim p.d.f. representing the population as a whole. If \( F_t(dy) \) does not depend on \( t \), we call this situation the time homogeneous case. (2.8) will be utilized later. Particularly in the proof of Theorem 5.1 this line of reasoning becomes important.

2.5 In principle the claims may be positive or negative. For example, a death may free the company of an obligation and increase the reserves. In practice a growing company will measure time in operational units

April 24, 1984
proportional to the total incoming premiums. It has then usually been assumed that in the absence of claims the reserves increase at a constant rate $\mu$. The Lundberg model (1.2) is a special case of (2.1) with $d = 1$, $\mu(t, X) \equiv \mu$, $\sigma(t, X) \equiv 0$, $\gamma(t, X, t) = -y$ and $\lambda(t, X, dy) = \lambda F(dy)$. In (2.1) the premium income has a random factor which makes the model more realistic due to uncertainties in the market. This stochastic component is a martingale in agreement with the usual economic equilibrium theory of perfect markets. For more on equilibrium in insurance markets, see e.g. Borch (1983). The expected income from premiums is $\int_{0}^{t} E(\mu(s, X)) ds$ in $[0, t]$, which reduces to $\mu t$ if $\mu(t, X) = \mu$.

2.6 The general point process component in (2.3) allows for several extensions of the model (1.2). Here we point out that the jump intensity $\lambda(t, X)$ may be a non-homogeneous $\Xi_t$-measurable random process depending on time, as well as on $X(s)$, $s \leq t$. The technical requirement of $\lambda(t, X)$ is predictability, which in application means that $\lambda(\cdot, X)$ is left-continuous and $\Xi_t$-measurable.

Furthermore, the conditional p.d.f. of the individual claims may depend on time, and the amounts claimed may be stochastically dependent. Such dependency will be reflected through the functionals $\gamma$ and $\lambda$. For more on modern stochastic calculus, see e.g. Elliot (1982).
3. THE PROBABILITY DISTRIBUTION OF THE STATE OF THE COMPANY:

Higher Order Approximations

3.1 In this section we establish an Edgeworth expansion of the p.d.f. $F(x; t) = P[X(t) \leq x]$, where $X(t)$ represents the state of the insurance company by time epoch $t$. $X(t)$ can be given either of the interpretations (a) or (b) outlined in section 2.2.

Consider the standardization $Z(t) = (X(t) - E(X(t)))/\sqrt{\text{var}(X(t))}$, and let $G(x; t) = P(Z(t) \leq x)$. Then our results will be related to the p.d.f. $G(x; t)$.

3.2 The standard normal p.d.f. is denoted by $\Phi(x)$ and $\varphi(x)$ is its density. A Berry-Esséen type result would, if possible, say something like $\sup_x |C(x; t) - \Phi(x)| \leq Ct^{-\frac{1}{2}}$, where $C$ is a constant depending upon the third moment $\mu_3$ given in (3.13) below. On a practical level, such a result is almost useless since the bound on the right is typically too large. However, it is theoretically useful as we shall see in section 4.

Edgeworth expansions provide better approximations: They specify functions $G_r(x; t)$ such that $\sup_x |C(x; t) - G_r(x; t)| = O(t^{-\frac{1}{2} + \epsilon})$, $r \geq 3$. Note that bounds of the form $O(t^{-\frac{1}{2} \log t})$ or $O(t^{-\frac{1}{2} + \epsilon})$, $\epsilon > 0$ are not considered good enough.

3.3 We return to the model of section 2 and make some simplifying assumptions. Interpretation (b) is utilized for convenience throughout this section. The minor changes necessary for interpretation (a) to hold, are pointed out in remark 4 after Theorem 3.1 below.

Suppose the stochastic integral equation (2.1) is of the form

$$X(t) = x_0 + \mu t + \sigma B(t) + \sum_{k=1}^{d} \sum_{n \geq 1} Y_{k,n} 1(T_{k,n} \leq t)$$  \hspace{1cm} (3.1)

with the intensity kernels

$$\lambda_k(t, X, dy) = \lambda_k f_k(dy), \hspace{1cm} k = 1, 2, ..., d$$  \hspace{1cm} (3.2)

where $\lambda_k$, $\mu$ and $\sigma$ are all positive constants. It is assumed that the process $B(t)$ is independent of the r.v.'s $Y_{k,n}$, the latter being all independent. For each $k$, $Y_{k,n}$ has p.d.f. $F_k(dy)$, for all $n$. The exact distribution of $X(t)$ is as follows:

$$P[X(t) \leq x] = F(x; t) = \Phi \left[ \frac{x - x_0 - \mu t}{\sigma t} \right].$$

April 24, 1984
\[ e^{-\lambda t} \sum_{n \geq 0} \frac{(\lambda t)^n}{n!} F_1^n(x) \ast \cdots \ast e^{-\lambda t} \sum_{n \geq 0} \frac{(\lambda t)^n}{n!} F_d^n(x) \]  
\text{(3.3)}

i.e., the convolution of the \((d+1)\) distributions indicated. As in (1.1) this expression quickly becomes more complicated as \(t\) grows.

3.4 Let \(W(t)\) be the Wiener process with drift starting in \(x_0\), given by
\[ W(t) = x_0 + \mu t + \sigma B(t) \]  
\text{(3.4)}

Its moment generating function (m.g.f.) is
\[ p(s) = E[e^{sW(t)}] = \exp\{s(x_0 + \mu t) + \frac{1}{2} t \sigma^2 s^2\} \]  
\text{(3.5)}

Also the m.g.f. of \(F_k(dy)\) is
\[ p_k(s) = \int_{-\infty}^{+\infty} e^{sy} F_k(dy), \quad k = 1, 2, \ldots, d \]  
\text{(3.6)}

These are assumed to exist and to be \(r\) times differentiable at \(s = 0\), \(r \geq 3\).

The m.g.f. of \(X(t)\) is
\[ E\{e^{sX(t)}\} = \exp\left\{ \sum_{k=1}^{d} \lambda_k t \left( p_k(s) - 1 \right) + s(x_0 + \mu t) + \frac{1}{2} \sigma^2 t s^2 \right\} \]  
\text{(3.7)}

Let
\[ q_t(s) = \log E\{e^{sX(t)}\} \]  
\text{(3.8)}

and define
\[ p_{k,l} = \int_{-\infty}^{+\infty} y^l F_k(dy), \quad k = 1, 2, \ldots, d \]  
\text{(3.9)}

i.e., the \(l\)th central moment of \(F_k(dy)\). From this we can compute the cumulants \(K_l\) of \(X\) as follows:
\[ K_l(X) = \left. \frac{\partial^l}{\partial s^l} q_t(s) \right|_{s=0}, \quad 1 \leq l \leq r \]  
\text{(3.10)}

which gives
\[
\begin{align*}
K_1(X) &= E[X(t)] = x_0 + \mu t + t \sum_{k=1}^{d} \lambda_k p_{k,1} \\
K_2(X) &= \text{var} X(t) = \sigma^2 t + t \sum_{k=1}^{d} \lambda_k p_{k,2} \\
K_i(X) &= t \sum_{k=1}^{d} \lambda_k p_{k,i}, \quad 3 \leq i \leq r
\end{align*}
\]  
\text{(3.11)}

April 24, 1984
We need the cumulants of the standardized process \( Z \), which are found using (3.11)

\[
\begin{align*}
K_1(Z) &= 0 \\
K_2(Z) &= 1 \\
K_l(Z) &= \mu_l t^{-\frac{1}{2}l+1} & 3 \leq l \leq r
\end{align*}
\] (3.12)

where

\[
\mu_l = \sum_{k=1}^{d_l} \lambda_k p_{k,l} \left[ \sum_{m=1}^{d_l} \lambda_m p_{m,2} + \sigma^2 \right]^{-l/2}
\] (3.13)

The characteristic function \( \psi_t(u) = E\{e^{iuZ(t)}\} \) plays an important role. Here

\[
\psi_t(u) = \exp \left[ \sum_{k=1}^{d} \lambda_k t \left[ f_k \left( \frac{u}{(\Sigma \lambda_k p_{k,2} + \sigma^2)^{1/2}} \right) - 1 \right] - iu t^{1/2} \frac{\Sigma \lambda_k p_{k,1}}{(\Sigma \lambda_k p_{k,2} + \sigma^2)^{1/2}} - \frac{1}{2} (u \sigma)^2 \frac{1}{(\Sigma \lambda_k p_{k,2} + \sigma^2)} \right]
\] (3.14)

where

\[
f_k(u) = \int_{-\infty}^{\infty} e^{ix} p_{k}(dx), & k = 1, 2, ..., d.
\]

The Edgeworth expansion of \( G(x,t) \) is denoted by \( G_r(x,t) \) and is given by

\[
G_r(x,t) = \Phi(x) + \varphi(x) \sum_{k=3}^{r} P_k(x) t^{-\frac{1}{2}k+1}
\] (3.15)

The polynomials \( P_k(x) \) are determined from the Gram-Charlier series, and the first terms are

\[
\begin{align*}
P_0(x) &= \frac{1}{6} \mu_3 H_3(x) \\
P_4(x) &= \frac{1}{24} \mu_4 H_3(x) + \frac{1}{72} \mu_5 H_5(x) \\
P_6(x) &= \frac{1}{120} \mu_5 H_4(x) + \frac{1}{144} \mu_6 H_6(x) + \frac{1}{1296} \mu_7 H_7(x)
\end{align*}
\] (3.16)

The \( \mu_k \)'s are given in (3.13) and \( H_k(x) \) are the Hermite polynomials, determined as follows:

\[
\begin{align*}
H_0(x) &= x^2 - 1 \\
H_1(x) &= x^3 - 3x \\
H_2(x) &= x^4 - 6x + 3 \\
H_{n+1}(x) &= xH_n(x) - nH_{n-1}(x), & n \geq 4
\end{align*}
\] (3.17)

April 24, 1984
One important property of these polynomials will be used in the sequel
\[ \int_{-\infty}^{\infty} e^{iu\varepsilon} d(H_{k}(x)\varphi(x)) = (iu)^{k} e^{-\frac{1}{2}u^{2}} \]  
(3.18)

By this property we find the Fourier transform \( \psi_{t,r}(u) \) of \( G_{r}(x,t) \) as follows:
\[ \psi_{t,r}(u) = e^{-\frac{1}{2}u^{2}} (1 + \sum_{i=3}^{r} R_{i}(iu)t^{-\frac{1}{2}i+1}) \]  
(3.19)

The polynomials \( R_{i}(\cdot) \) are determined from \( P_{i}(x) \) using (3.18), and the first terms are
\[
\begin{align*}
R_{3}(iu) &= \frac{1}{6} \mu_{3}(iu)^{2} \\
R_{4}(iu) &= \frac{1}{24} \mu_{4}(iu)^{3} + \mu_{5}(iu)^{5} \\
R_{5}(iu) &= \frac{1}{120} \mu_{5}(iu)^{4} + \frac{1}{144} \mu_{3} \mu_{4}(iu)^{6} + \frac{1}{1296} \mu_{3}^{2}(iu)^{6} \\
& \quad \vdots
\end{align*}
\]  
(3.20)

We can now state the main result in this section.

**Theorem 3.1.** Suppose the following conditions are satisfied:

(i) The moments \( P_{k,1}, P_{k,2}, \ldots, P_{k,r}, \ k = 1, 2, \ldots, d, \ r \geq 3 \) exist and \( \sigma^{2} < \infty \).

(ii) The characteristic functions \( f_{k}(u) \) of the conditional individual claim distributions satisfy \( \limsup_{|u| \to \infty} |f_{k}(u)| < 1, \ k = 1, 2, \ldots, d \).

Then, as \( t \to \infty \),
\[ G(x,t) = \Phi(x) + \varphi(x) \sum_{k=3}^{r} P_{k}(x)t^{-\frac{1}{2}k+1} + o(t^{-\frac{1}{2}r+1}) \]

uniformly in \( x \).

**3.5 Before we give the proof, note the following:**

1. The polynomials \( P_{k}(x) \) can be computed without knowledge of the distributions \( F_{k}(dy) \) or the distribution of \( W \), only the moments \( P_{k,l}, \ k = 1, 2, \ldots, d, \ l = 1, 2, \ldots, r \) and \( \sigma^{2} \) are needed. Also \( P_{k}(x) \) do not dependent on \( t \) or \( r \).

2. \( \mu_{3} \) corrects for skewness, whereas \( \mu_{4} \) is the kurtosis correction term.

3. Condition (ii) is called Cramér's condition \( (d = 1) \). It appears to be a weak requirement and for \( F_{k}(dx) \) non-lattice, it is so mild as to be trivially satisfied in most cases.
The result of Theorem 3.1 can be adapted to interpretation (a) in section 2 by substituting $\mu_k$ by $(-1)^k \mu_k$ and $\mu + \sum_{k=1}^d \lambda_k p_{k,1}$ by $\mu - \sum_{k=1}^d \lambda_k p_{k,1}$. Otherwise, no further changes are required. This follows from elementary properties of cumulants.

3.6 Proof of Theorem 3.1 Esséen's smoothing lemma will be used here. It states that

$$\sup_x |G(x,t) - G_r(x,t)| \leq \frac{1}{\pi} \int_U \frac{\psi_t(u) - \psi_{t,r}(u)}{u} \, du + \frac{CM}{U}$$  \hspace{1cm} (3.21)

$M > 0$ is a constant such that $|\frac{\partial}{\partial x} G_r(x,t)| \leq M$, $C > 0$ is some constant and $U > 0$. We choose $U = at^{\frac{1}{2}r-1}$ for some positive constant $a$, and focus attention on the integral in (3.21). First we divide the domain of integration into the following two sections:

$$E_{1,t} = \{ u : |u| \leq \delta(t \Sigma \lambda_k p_{k,2} + \sigma^2)^{1/2} \}$$

for some $\delta > 0$ and

$$E_{2,t} = \{ u : \delta(t \Sigma \lambda_k p_{k,2} + \sigma^2)^{1/2} \leq |u| \leq at^{\frac{1}{2}r-1} \}$$

We proceed in two steps:
(a) Consider

$$\int_{E_{1,t}} \frac{\psi_t(u) - \psi_{t,r}(u)}{u} \, du$$  \hspace{1cm} (3.22)

By a Taylor series expansion of the characteristic function $\psi_t(u)$ using (3.12) we get

$$\log \psi_t(u) = -\frac{1}{2} u^2 + \sum_{k=3}^r (ivu)^k \mu_k t^{-\frac{1}{2}k+1} + o(t^{-\frac{1}{2}r+1} |u|^r)$$  \hspace{1cm} (3.23)

By exponentiating (3.23) for $u \in E_{1,t}$, it can be shown that

$$\psi_t(u) = \psi_{t,r}(u) + o(t^{-\frac{1}{2}r+1} |u| e^{-\frac{1}{4}u^2})$$

where $\psi_{t,r}$ is given in (3.19). Hence it follows that the integral in (3.21) is $o(t^{-\frac{1}{2}r+1})$.

(b) Next, consider

$$\int_{E_{2,t}} \frac{\psi_t(u) - \psi_{t,r}(u)}{u} \, du$$  \hspace{1cm} (3.24)
Clearly, this integral is not larger than
\[ \int_{\xi_{2,t}} \left| \frac{\psi_t(u)}{u} \right| \, du + \int_{\xi_{2,t}} \left| \frac{\psi_{t+\xi}(u)}{u} \right| \, du \]

Since the exponential function dominates any power of \( t \), the last integral is \( o(t^{-k}) \) for every \( k \). Hence it suffices to focus attention on the first integral.

\[ \int_{\xi_{2,t}} \left| \frac{\psi_t(u)}{u} \right| \, du \]

\[ = \int_{\xi_{2,t}} \frac{1}{|u|} \exp \left\{ \sum_{k=1}^{d} \lambda_k \left[ f_k \left( \frac{u}{(t \Sigma_{i=1}^{d} \lambda_i p_{i,2} + \sigma^2)^{1/2}} \right) - 1 \right] - \frac{1}{2} \left( \sigma u \right)^2 \frac{1}{(\Sigma_{k=1}^{d} \lambda_k p_{k,2} + \sigma^2)^{1/2}} \right\} \, du \]

\[ \leq \int_{\xi_{2,t}} \frac{1}{|u|} \exp \left\{ \sum_{k=1}^{d} \lambda_k \left[ f_k \left( \frac{u}{(t \Sigma_{i=1}^{d} \lambda_i p_{i,2} + \sigma^2)^{1/2}} \right) - 1 \right] \right\} \, du \]

\[ \leq 2 \exp \left( (\rho_t - 1) t \sum_{k=1}^{d} \lambda_k \right) \left\{ \frac{1}{2} \left( \frac{\sigma t}{\delta (\Sigma_{k=1}^{d} \lambda_k p_{k,2} + \sigma^2)^{1/2}} \right) - 1 \right\} \]

where \( \rho_t = \sup_{u \in \xi_{2,t}} f_k \left( \frac{u}{(t \Sigma_{i=1}^{d} \lambda_i p_{i,2} + \sigma^2)^{1/2}} \right) \).

By condition (ii) \( \lim_{t \to \infty} \rho_t < 1 \), the last term decreases to zero faster than any power of \( 1/t \) as \( t \) increases. This completes the proof. \( \square \)

3.7 Expansions for densities may be of interest. In the i.i.d. case the requirement is that

\[ \int_{-\infty}^{\infty} |f(u)|^v \, du < \infty, \quad v \geq 1 \]  

(Feller (1971)). In the present case this assumption does not work. The characteristic function of \( X(t) \) is \( (\sigma^2 = 0, d = 1) \) \( \mathbb{E}[f^{(n)}(t)] \geq f^{(n)}(t) \) by Jensen's inequality. Here strict inequality holds, which only tells us that the proof is more difficult than for the i.i.d. case with non-random \( n \). In fact (3.25) does not imply that \( \int_{-\infty}^{\infty} |\exp(\lambda t \left[ \frac{u}{(t \lambda p_{i,2})^{1/2}} \right] - 1)| \, du \) converges to zero as \( t \) increases. This last integral fails to converge in \( u \), for fixed \( t \), since the integrand does not tend to zero as \( |u| \) increases. One may say that (3.25) is irrelevant for this convergence. It follows that it cannot be shown by the usual techniques that the Fourier norm of the density minus its Edgeworth expansion converges to zero at any rate. In this situation Esséen's smoothing lemma proves to be a powerful tool. Because we could choose \( U \) to be finite for \( t \) fixed, we avoided integrability problems of this kind in the second part of the proof.

April 24, 1984
It seems intuitively reasonable that an expansion for densities should be valid in many cases of practical concern. It would be interesting to know the conditions that are necessary for such expansions to hold true to order $o(t^{-r+1})$ uniformly in $x$. 
4. APPROXIMATIONS IN THE TAILS OF THE DISTRIBUTION

4.1 It is known that Edgeworth expansions perform poorly in the tails of the distribution: They can exceed one or fall below zero. For large $x$, both $G(x;t)$ and $\Phi(x)$ are close to unity, but they may not be of the same order of magnitude. An estimate is needed of the relative error in approximating $1-G(x;t)$ by $1-\Phi(x)$, i.e. we need results of the form

$$\frac{1-G(x;t)}{1-\Phi(x)} \to 1$$

where both $x$ and $t$ tend to infinity. In this section we prove an analogue of the classical result for the i.i.d. case which asserts that (4.1) is true if $x$ varies with $n$ in such a way that $x \cdot n^{-\frac{1}{6}} \to 0$ (Feller, (1971)).

4.2 Consider the m.g.f. of $F_k(dx)$ given in (3.6). The necessary condition is that these integrals exist for all $s$ in some interval around the origin.

**Theorem 4.1.** Suppose

(i) $E X(t) \geq 0$ for all $t \geq 0$

(ii) $\int_{-\infty}^{\infty} e^{sx} F_k(dx)$ exists in some interval $|s| < s_0$, $k = 1, 2, \ldots, d$.

If $x$ varies with $t$ in such a way that as $x \to \infty$, $x = o(t^{1/6})$, then

$$\frac{1-G(x;t)}{1-\Phi(x)} = 1 + O\left(\frac{x^3}{\sqrt{t}}\right).$$

4.3 Remarks:

1 Assumption (i) is naturally satisfied when applied to the insurance industry:
   In case (a) of section 2, a violation implies bankruptcy with probability one. In case (b), (i) obviously is satisfied.

2 The result implies that good approximations to $G(x;t)$ can be achieved for certain large values of $x: x = o(t^{1/6})$. Furthermore, it gives an estimate of the relative error committed in using this approximation.

3 This is a large deviation result. By changing $x$ into $-x$, we obtain a dual result for the left tail.

4.4 Proof of Theorem 4.1 From (3.7) and (3.8) we get

$$q_t(s) = \log E\{e^{sx(t)}\} = \sum_{k=1}^{d} \lambda_k t (p_k(s) - 1) + s(x_0 + \mu t) + \frac{1}{2} \sigma^2 t s^2.$$

It follows from assumption (ii) that $q_t(s)$ is analytic in a neighborhood of
the origin. In order to utilize this property in an elegant way, we use the technique of associated distributions. With the distribution \( F(x;t) = P[X(t) \leq x] \) we associate a new probability distribution \( H_t(x) \) such that
\[
H_t(dx) = e^{-q_t(u)}e^{ux}F(dx;t) \tag{4.2}
\]
The parameter \( u \) is chosen in the interval where \( q_t(s) \) is analytic. The m.g.f. of \( H_t(dx) \) is
\[
h_t(s) = \frac{q_t(u+s)}{q_t(u)} \tag{4.3}
\]
In particular it follows from (4.3) that \( H_t(dx) \) has expected value \( q_t'(u) \) and variance \( q_t''(u) \).

As in section 3, \( C(x;t) \) is the p.d.f. of the normalized process \( Z(t) \).

From (4.2) it is seen that
\[
1 - C(x;t) = e^{q_t(u)} \int_a(x,t) e^{-uy}H_t(dy) \tag{4.4}
\]

with
\[
a(x,t) = x(t \Sigma \lambda_k p_{k,2} + \sigma^2)^{1/2} + (x_0 + t \Sigma \lambda_k p_{k,1} + \mu)
\]

In view of the central limit theorem, we want to replace \( H_t(dy) \) in (4.4) by the corresponding normal distribution with expectation \( q_t'(u) \) and variance \( q_t''(u) \). We denote this latter distribution by \( \Phi_u(dx) \). The error committed will be small if \( a(x,t) \) is close to the expected value of \( H_t(dx) \). This motivates the choice
\[
q_t'(u) = a(x,t) \tag{4.5}
\]
In a neighborhood of the origin, \( q_t(s) \) is strictly increasing, since by assumption (i) \( q_t'(0) = E(X(t)) \geq 0 \). By the implicit function theorem, (4.5) therefore establishes a one-to-one correspondence between the variables \( u \) and \( x \), as long as \( u \) and \( x/\sqrt{t} \) are restricted to a suitable neighborhood of the origin. We may consider each variable to be an analytic function of the other. From (4.5) we get that
\[
u \sim \frac{x t^{-1/2}}{(\Sigma \lambda_k p_{k,2} + \sigma^2)} \text{ if } xt^{-1/2} \to 0. \tag{4.6}
\]
We now proceed in two steps:

(a) We denote a quantity \( R_u \) obtained from the right side of (4.4) by replacing \( H_t(dy) \) by \( \Phi_u(dy) \). The standard substitution \( y = q_t'(u) + z \sqrt{q_t''(u)} \) yields
\[
R_u = \exp[q_t(u) - uq_t'(u) + \frac{1}{2} u^2 q_t''(u)] \tag{4.7}
\]
April 24, 1984
The exponent can be shown to be \( O(u^3) \), i.e., it starts with a cubic term. Thus

\[
R_u = [1 - \Phi(u \sqrt{q''(u)})][1 + O(tu^3)]
\]

If \( tu^3 \to 0 \), or \( x = o(t^{1/6}) \) by use of (4.6), \( R_u \) may be rewritten as follows:

\[
R_u = [1 - \Phi(y)][1 + O\left(\frac{x^3}{\sqrt{t}}\right)] \tag{4.8}
\]

where \( y = u \sqrt{q''(u)} \). It remains for us to show that in (4.8) \( y \) may be replaced by \( x \). If we use a Taylor series expansion of the square root function, terms cancel neatly and the power series for \( (y - x) / \sqrt{t} \) can be shown to start with a cubic term. Accordingly,

\[
|y - x| = O(\sqrt{t} u^3) = O\left(\frac{x^3}{\sqrt{t}}\right) \tag{4.9}
\]

Also,

\[
\frac{\varphi(z)}{1 - \Phi(z)} \sim z \quad \text{as} \quad z \to \infty \tag{4.10}
\]

Integrating between \( x \) and \( y \), we get as \( x \to \infty \)

\[
\left| \log \frac{1 - \Phi(y)}{1 - \Phi(x)} \right| = O(x \cdot |y - x|) = O\left(\frac{x^4}{t}\right) \tag{4.11}
\]

Hence

\[
\frac{1 - \Phi(y)}{1 - \Phi(x)} = 1 + O\left(\frac{x^4}{t}\right) \tag{4.12}
\]

By substituting (4.12) into (4.8), we get, as \( x \to \infty \), \( x = o(t^{1/6}) \)

\[
R_u = [1 - \Phi(x)][1 + O\left(\frac{x^3}{\sqrt{t}}\right)] \tag{4.13}
\]

(b) From the definition of \( R_u \) we have

\[
R_u = e^{q(u)} \int_{q_i(u)} e^{-uy} \Phi_u(dy). \tag{4.14}
\]

We want to find the error committed by the replacement of \( H_i(dy) \) by \( \Phi_u(dy) \) in (4.4). By a version of the Berry-Esséen theorem, which in the present case follows from Theorem 3.1 with \( r = 3 \), we have

\[
|H_i(y) - \Phi_u(y)| < C_u t^{-1/2}. \tag{4.14}
\]

Here \( C_u < \infty \) if \( H_i(dx) \) has a finite third absolute moment. This is implied by the analytic behavior of \( q_i(s) \) for \( s = u \), since the third moment equals \( q''''(u)/q_i(u) \). Now
\begin{align*}
(1 - C(x; t) - R_u) &= \int_{q'_t(u)}^{\infty} e^{-u\gamma} (\tilde{H}_t(x, y) - \tilde{\Phi}_u(x, y)) \, du \\
&\quad - e^{q'_t(u)} \int_{q'_t(u)}^{\infty} e^{-uq'_t(u)} (\tilde{H}_t(q'_t(u)) - \tilde{\Phi}_u(q'_t(u))) \\
&\quad + u \int_{q'_t(u)}^{\infty} (\tilde{H}_t(u) - \tilde{\Phi}_u(y)) e^{-u\gamma} \, dy \]

by an integration by parts. Using (4.14) we get

\begin{align}
1 - C(x; t) - R_u < \frac{2C_\gamma}{\sqrt{t}} e^{q(u) - uq'_t(u)} & \quad (4.15) \end{align}

Also

\begin{align*}
R_u &= e^{\frac{1}{2\gamma^2} (1 - \Phi(u))} e^{q(u) - uq'_t(u)}

\end{align*}

which follows directly from (4.7). Furthermore, from (4.10)

\begin{align*}
R_u \sim \frac{1}{x} e^{q(u) - uq'_t(u)} & \quad \text{as } x \to \infty

\end{align*}

Accordingly,

\begin{align*}
1 - C(x; t) &= R_u (1 + O(\frac{x}{\sqrt{t}}))

\end{align*}

By this and (4.13) the proof of the theorem follows.
5. COLLECTIVE RISK IN INSURANCE

The Probability of Ruin

5.1 Several measures of risk may be used in the insurance industry. One that has been proposed is the probability \( R(x_0) \) that no ruin will ever occur. We will consider this measure in the present section, using the same model as in section 3 and interpretation (a) from section 2, i.e.

\[
X(t) = x_0 + \mu t + \sigma B(t) - V(t)
\]

where

\[
V(t) = \sum_{k=1}^{d} \sum_{n \geq 1} Y_{k,n} 1(T_{k,n} \leq t)
\]

\( X(t) \) represents the company's total reserve at time epoch \( t \), \( x_0 \) is the security reserve at epoch 0 and \( \mu \) equals the expected rate of income from premiums, where the local variance is \( \sigma^2 \). \( V(t) \) stands for the accumulated amount of claims by time \( t \), and ruin by time \( t \) is the event \( A_t \), where

\[
A_t = \{ V(t) > x_0 + \mu t + \sigma B(t) \}
\]

Also,

\[
R(x_0) = \mathbb{P} \left( \bigcap_{t \geq 0} A_t^c \right)
\]

Since both \( B(t) \) and \( V(t) \) are separable stochastic processes, the uncountable intersection of the events \( A_t^c \) is again an event, and hence it has a probability. We shall argue formally below that \( R(x_0) \) must be a non-increasing solution of the functional equation (5.8).

5.2 This measure of risk should not be interpreted literally as a probability of no bankruptcy. Here, the profit of the company is modeled too simplistically for such an interpretation to hold. If \( \mu \), \( R(x_0) \) is small, we should not automatically conclude that the company is in danger. We might, however, use

\[
R(x_0) = R(x_0, \mu, \sigma, F)
\]

to compare different options for the company: Suppose the fluctuations in the incoming premiums are reduced from \( \sigma \) to \( \sigma - \Delta \). We might ask, by how much could the premiums be reduced without changing the risk? In other words, we want to find \( \delta \) such that

\[
R(x_0, \mu, \sigma, F) = R(x_0, \mu - \delta, \sigma - \Delta, F),
\]

Now suppose the expected premiums were reduced from \( \mu \) to \( \mu - \delta \). We might then ask the following: By how much should the security reserve be
raised in order for the risk to remain unchanged? Here we want to find \( \Delta \) such that \( R(x_0 + \Delta, \mu, \sigma, F) = R(x_0, \mu - \delta, \sigma, F) \). Finally, suppose the company is reinsured so that the individual conditional claim distribution \( F \) is changed to \( C \). We might ask a third question: How much could the expected premiums be reduced without changing the risk? In terms of the function \( R(\cdot) \) we want to find \( \delta \) such that \( R(x_0, \mu - \delta, \sigma, C) = R(x_0, \mu, \sigma, F) \).

Naturally, we can only raise these questions if we assume that \( R(x_0) \) is an acceptable measure of risk.

One obvious weakness with the above model of the profit is that it does not take into account a dividend policy. Such a policy basically implies that local accumulated profits beyond a certain level has lower utility than a paid-out dividend. One of the first to point this out was De Finetti (1957). See also Borch (1984). However, with this in mind and used with care, the above interpretation of \( R(\cdot) \) may be useful.

5.3 Let \( T = \) time epoch of the first claim and \( x = \) magnitude of this claim. We only know a claim was made at time \( T \), but we do not know which type \( k \) it was, \( k \in \{1, 2, \ldots, d\} \). In order to find the probability in question, we must first prove some lemmas. Since \( B(t) \) is a random process, clearly a ruin may occur for \( t \in (0, T) \). In Lemma 5.1 below we find the distribution of the first time a Brownian motion with positive drift starting at \( x_0 > 0 \), hits zero. This distribution must be defective, since there is a positive probability that such a process never reaches zero.

Lemma 5.1. Let \( W(t) \) be a Brownian motion with negative drift parameter \( c < 0 \). Let \( z > W(0) = x_0 \) and let \( T_z \) be the first time the process reaches level \( z \). Given that \( W(0) = x_0 \), \( T_z \) has the defective probability density function

\[
f(t; x_0, z) = \frac{z - x_0}{\sigma \sqrt{2\pi t}} e^{-\frac{(x-x_0)^2}{2\sigma^2}} e^{-\frac{(z-x_0+ct)^2}{2\sigma^2}}, \quad t > 0.
\]

(5.5)

**Proof.** We use the known result of the proper probability density function in the case \( c > 0 \) (e.g., Karlin and Taylor (1975), Theorem 5.3), and the known form of the characteristic function \( \psi(u) \) of \( T_z \) for arbitrary \( c \). By this and the inversion formula we get

\[
f(t; x_0, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itu} \psi(u) du
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itu} e^{-\frac{(z-x_0)^2}{2\sigma^2}} e^{-\frac{(z-x_0+ct)^2}{2\sigma^2}} du
\]

\[
= e^{-\frac{(z-x_0)\sigma}{\sigma^2}} \cdot \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itu} e^{-\frac{(z-x_0)^2}{2\sigma^2} + \frac{ict}{\sigma^2}} du
\]

April 24, 1984
\[ \frac{2(z-x_0) |c|}{\sigma^2} \cdot \frac{z-x_0}{\sigma \sqrt{2 \pi t^3}} e^{-(z-x_0-c \cdot t)^2/2\sigma^2 t} \]

The last inversion is based on the known formula for \( c > 0 \).

**Corollary 5.1.** The probability that \( W(t) \) in Lemma 5.1 never hits \( z \), given that it starts in \( x_0 \), equals

\[ 1 - \exp\{-2(z-x_0) |c|/\sigma^2\} \]

**Proof.** For \( c > 0 \), clearly

\[ \int_0^\infty \frac{z-x_0}{\sigma \sqrt{2 \pi t^3}} \exp\{-\frac{1}{2\sigma^2 t}(z-x_0-ct)^2\} dt = 1. \]

The result now follows from Lemma 5.1.

5.4 Returning to \( R(x_0) \), we can now prove the following theorem:

**Theorem 5.1.** Suppose

\[ F(dy) = F_1(dy) = \sum_{k=1}^d \lambda_k(t) P_k(dy)/\lambda(t) \]

where \( \lambda = \lambda(t) = \sum_{k=1}^d \lambda_k(t) \) does not depend on \( t \). Then the probability \( R(x_0) \) that no ruin will ever occur satisfies the following integral equation:

\[ R(x_0) = \frac{\lambda}{\mu} \int_{x_0}^\infty \frac{1}{\mu} \left[ \frac{\mu}{2\pi(s-x_0)} \right]^{1/2} e^{-\frac{\mu(v-s)^2}{2\sigma^2(s-x_0)}} \]

\[ \int_{-\infty}^{\infty} R(v-x) F(dx) dv \left\{ \int_{\infty}^{x} \frac{(z_0 + \mu t)^2}{2\sigma^2 t} dt + 1 - e^{-\frac{2\pi \mu t}{\sigma \sqrt{2 \pi t^3}}} \right\} ds \quad (5.6) \]

**Proof.** Let \( P_{z,T}(\cdot) \) be the conditional probability measure given that the first claim is of size \( z \) at time \( T \). Also, let \( T_0 \) be the first time \( W(t) \) in (3.4) hits zero: \( W(t) = x_0 + \mu t + \sigma B(t) \). First notice that \( P_{z,T} \) (no ruin for \( t \leq T \)) = \( P_{z,T}[W(T) \equiv z | W(t) > 0 \text{ for all } t \in [0,T]) \cdot P_{z,T}(T_0 > T) \).

The last equality above follows from the path continuity of \( W(t) \) and from the strong Markov property of \( B(t) \).

April 24, 1984
For \( t > T \), the increments must satisfy \( V(t) - x \leq x_0 + \mu t + \sigma B(t) - x \) in order to avoid ruin. Since these increments are independent of the past, the conditional probability of this last event is \( R(W(T) - x) \). Now we use the fact that \( B(t) \) is normally distributed, we use Lemma 5.1 and Corollary 5.1 to compute \( P_{x,T}(T_0 > T) \), and we take advantage of the remarks made in section 2.4 regarding the interpretation of \( F_t(dx) \). Integrating over all possible \( T \), \( B(T) \) and \( x \), we get

\[
R(x_0) = \int_0^\infty e^{-\lambda y} \int_{(x_0 + \mu t) - 2\sigma^2 \sqrt{2\pi t}} \int R(x_0 + \mu t + \sigma b - x) F(dx)db
dt
\]

By the substitution \( s = x_0 + \mu t \) in (5.7) we obtain the following:

\[
R(x_0) = \frac{\lambda}{\mu} \int_0^\infty e^{-\lambda y} \int_{s-x_0}^\infty e^{-\left(\frac{\mu}{2\sigma^2}\right)\left(\frac{y^2}{s-x_0}\right)^2} ds
\]

Finally, by the substitution \( y = (y-s) / \sigma \) in (5.8) we find (5.6).

5.5 \( R(x_0) \) (satisfying (5.6)) can not be differentiated with respect to \( x_0 \). Hence, in general we cannot establish an integro-differential equation for \( R(x_0) \). Recall also the simpler case in section 1, where \( \sigma^2 = 0 \).

By letting \( \sigma \to 0 \) in (5.6) and by using the theory of generalized function, we formally obtain (1.3).

The solution \( R(x_0, \sigma^2) \) to (5.6) can be majorized by \( R(x_0) \), which is a solution to (1.3). Wiener-Hopf techniques may prove to be useful in handling equation (5.6).

April 24, 1984
Acknowledgement. I would like to thank P. Guttorp and S. Hubert for helpful suggestions. I would also like to thank W. Carmichael for drawing my attention to higher order approximations.

REFERENCES


April 24, 1984


April 24, 1984