NON-NORMALITY OF A CLASS OF RANDOM VARIABLES

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Let $\Phi$ be a non-decreasing, non-positive, left continuous function defined on $(0, \infty)$ for which

$$\int_{-\infty}^{\infty} \Phi^2(s) \, ds < \infty \quad \text{for all } \varepsilon > 0. \quad (1)$$

Thus $b \Phi^2(b) \to 0$ as $b \to \infty$, since $(b/2) \Phi^2(b) \leq \int_{b/2}^{b} \Phi^2(s) \, ds \to 0$ as $b \to \infty$. Let $\ln(t)$, $0 \leq t < \infty$, be a right continuous Poisson process with rate 1 and corresponding jump times $S_1, S_2, \ldots$. Set for all $-\infty < p < \infty$ and $0 < u, v < \infty$

$$h_p(u, v) = \int_u^v w^p dw / v^p = \begin{cases} (v^{p+1} - u^{p+1}) / ((p+1)v^p), & p \neq -1 \\ v \log(v/u), & p = -1. \end{cases}$$

For any $-\infty < p < \infty$ and integer $k \geq 1$ consider the random variable

$$V_{p, k} = \int_{S_k} h_p(\ln(s), s) \, d\Phi(s) + \int_{k} h_p(k, s) \, d\Phi(s).$$

This class of rv arises as natural limits for $L$-statistics in Mason and Shorack (1988). In that paper the following theorem is required.

**THEOREM 1.** The random variable $V_{p, k}$ is never a non-degenerate normal random variable for any choice of $-\infty < p < \infty$ and integer $k \geq 1$.

**PROOF.** Roughly, the idea of the proof is to show that if $p \leq -\frac{1}{2}$, then for any integer $k \geq 1$, $V_{p, k}$ has a lighter left tail than any non-degenerate normal random variable, whereas if $p > -\frac{1}{2}$ and $\Phi$ is not identically equal to zero, then $V_{p, k}$ has a heavier left tail than any non-degenerate normal random variable. This of course proves that $V_{p, k}$ cannot be non-degenerate normal for any choice of $p$ and integer $k \geq 1$. For convenient reference later on we begin by collecting a number of facts essential to the main body of the proof.

For all $-\infty < p < \infty$ and $0 < \lambda < \infty$ let

$$\zeta_p(\lambda) = \begin{cases} (1 - \lambda^{p+1}) / (p+1), & p \neq -1 \\ -\log \lambda, & p = -1. \end{cases}$$
Observe that $\zeta_\rho$ is strictly decreasing,

(2) \[ \zeta_\rho(u, v) = v^{-1} h_\rho(u, v), \]

and for any $-\infty < \rho < \infty$ and $0 < u, v, b < \infty$ we have trivially that

(3) \[ h_\rho(u, v + b) = (v + b) \zeta_\rho(u/(v+b)) = b \zeta_\rho(u/(v+b)) + v \zeta_\rho(u/(v+b)). \]

Notice that for each fixed $v$,

(4) \[ h_\rho(u, v) \text{ is a strictly decreasing function of } u, \]

and for $0 < u, v < \infty$ (using a two term Taylor expansion)

(5) \[ h_\rho(u, v) \leq (\geq) h_0(u, v) \text{ when } \rho \geq 0 \text{ (when } \rho \leq 0). \]

Also since $\zeta_\rho$ is a strictly decreasing function of $\lambda$, we have from (3) that for any $-\infty < \rho < \infty$ and $0 < u, v, b < \infty$

(6) \[ h_\rho(u, v + b) \geq b \zeta_\rho(u/(v+b)) + h_\rho(u, v). \]

We shall require the following three Poisson distribution inequalities.

**INEQUALITY 1.** For every $\lambda > 0$ we have

\[ P(N(\lambda) \geq x \lambda) \leq \exp(\lambda(x - 1 - x \log x)), \text{ if } x > 1 \]
\[ \leq \exp(-\lambda x \log x)/2, \text{ if } x \geq e^2. \]

**PROOF.** Use elementary moment generating function techniques. See, for example, Shorack and Wellner (1986, p. 486).

**INEQUALITY 2.** For every $\lambda > 0$ there exists a constant $0 < K(\lambda) < \infty$ such that for all $z \geq 1$ we have

\[ P(N(\lambda) \geq z) \geq K(\lambda) z^{-\epsilon} \exp(-z(\log z - \log(\lambda e))). \]

**PROOF.** The proof follows easily from Stirling's formula.

**INEQUALITY 3.** For all $0 < \lambda < 1$ and $z > 0$ we have
(9) \[ P \left( \sup_{t \geq 0} \{ N(t) - t/\lambda \} > z \right) \leq D(\lambda) \exp(z \log \lambda), \]

where \( 0 < D(\lambda) = (1 - \lambda e^{-\lambda + 1})^{-1} < \infty. \)

**Proof.** From Dwass (1974), e.g. Shorack and Wellner (1986, p. 392),

(10) \[ P \left( \sup_{t \geq 0} \{ N(t) - t/\lambda \} > z \right) = \sum_{n > z} \frac{(n - z)^n}{n!} (\lambda e^{-\lambda})^n e^{- z(1 - \lambda)}. \]

Now by using \( n! > (n/e)^n, (1 - z/n)^n \leq e^{-z} \) for \( 0 < z < n \) and \( \lambda \exp(1 - \lambda) < 1 \) for \( 0 < \lambda < 1 \), we see that the right side of (10) is bounded above by

\[ \sum_{n > z} (\lambda e^{-\lambda + 1})^n \exp((\lambda - 1)z) (1 - \lambda) \leq D(\lambda). \]

For any \(-\infty < \rho < \infty\) and integer \( k \geq 1 \), we can write

(11) \[ V_{\rho,k} = \int_k^\infty \rho \{ N(s) \vee k, s \} d\Phi(s) + \int_k^k \rho \{ N(s) \vee k, k \} (k/s)^{\rho} d\Phi(s) \]

\[ = V_{\rho,k}^{(1)} + V_{\rho,k}^{(2)}. \]

For any \( x \geq 0 \), set

\[ V_{\rho,k}(x) = \int_x^\infty \rho \{ N(s-x) \vee k, s \} d\Phi(s) + \int_x^x \rho \{ k, s \} d\Phi(s). \]

Note from the original representation for \( V_{\rho,k} \) that

(12) conditioned on \( S_k = x \), \( V_{\rho,k} = V_{\rho,k}(x). \)

We shall first prove the theorem for the harder case \( \rho \leq -\frac{1}{2} \). This will require a number of lemmas.

**Lemma 1.** For all \( x, b \geq 0, -\infty < \rho < \infty \) and integers \( k \geq 1 \) we have

(13) \[ P(V_{\rho,k}(x) > z) \leq P(V_{\rho,k}(x+b) > z). \]

**Proof.** By (4) we have both

\[ \int_k^x \rho \{ k, s \} d\Phi(s) + \int_k^x \rho \{ N(s-x) \vee k, s \} d\Phi(s) \leq \int_k^{x+b} \rho \{ k, s \} d\Phi(s) \]
and conditioned on $\text{IN}(b) = m$,

$$\int_{x+b}^{\infty} h_p (\text{IN}(s-x) + k, s) \, d\Phi(s) \equiv \int_{x+b}^{\infty} h_p (\text{IN}(s-x-b) + m + k, s) \, d\Phi(s)$$

$$\leq \int_{x+b}^{\infty} h_p (\text{IN}(s-x-b) + k, s) \, d\Phi(s).$$

By the above two inequalities we have (13). □

**Lemma 2.** For all $x$ and $z$

$$P \left( V_{\rho,k}^{(1)} > z \mid S_k = x \right) \leq P \left( V_{\rho,k}(x+k) > z \right).$$

**Proof.** Since $V_{\rho,k}^{(1)} = V_{\rho,k}$ when $S_k \geq k$, by Lemma 1 we need only consider the case $0 < x < k$.

Given $S_k = x$ with $0 < x < k$,

$$V_{\rho,k}^{(1)} = \int_{k}^{\infty} h_p (\text{IN}(s), s) \, d\Phi(s);$$

and this conditional distribution is the same as the distribution of

$$\int_{k}^{\infty} h_p (\text{IN}(s-x) + k, s) \, d\Phi(s).$$

By (4), the latter rv is

$$\leq \int_{k}^{\infty} h_p (\text{IN}(s-k) + k, s) \, d\Phi(s) = V_{\rho,k}(k).$$

Thus for $0 < x < k$

$$P \left( V_{\rho,k}^{(1)} > z \mid S_k = x \right) \leq P \left( V_{\rho,k}(k) > z \right),$$

which by Lemma 1 is less than or equal to $P \left( V_{\rho,k}(k+x) > z \right)$. □

**Lemma 3.** For each $k \geq 1$ there exists a constant $0 < c_k < \infty$ such that for all $-\infty < \rho, z < \infty$

$$P \left( V_{\rho,k}^{(1)} > z \right) \leq c_k P \left( V_{\rho,k} > z \right).$$
PROOF. Let $f_k$ denote the density of $S_k$,

\[ P\left(V^{(1)}_{p,k} > z\right) = \int_0^\infty P\left(V^{(1)}_{p,k} > z \mid S_k = x\right)f_k(x)\,dx , \]

which by Lemma 2 is

\[ \leq c_k \int_0^\infty P\left(V^{(1)}_{p,k} > z\right)f_k(x)\,dx , \]

where

\[ c_k = \sup_{x \geq 0} \frac{f_k(x)}{f_k(x + k)} . \]

By (12) the right side of the last inequality equals $c_k P(V_{p,k} > z)$. □

For any $b > 0$, set $\Phi_b(s)$ equal $\Phi(b)$, $\Phi(s)$ according as $0 < s \leq b$, $b < s$, and let

\[ U(b) = \int_{\delta_b}^s (s - \ln(s))\,d\Phi_b(s) + \int_1^s d\Phi_b(s) + \Phi(b) . \]

**Lem. 4.** For all $b \geq 1$ and $0 < t < -\gamma/\Phi(b)$, where $\gamma > 0$,

(16) \[ E \exp(-t \, U(b)) \leq \exp\left(-t \Phi(b) + \frac{t^2}{2} f(b) e^t\right) , \]

where

\[ f(b) = \int_b^\infty \Phi_0^2(s)\,ds = b \Phi_0^2(b) + \int_b^\infty \Phi_0^2(s)\,ds . \]

PROOF. Arguing as in the proof of Theorem 3 of CHM [1989], one obtains

\[ \log E \exp(-t \, U(b)) = \int_0^\infty \left\{ \exp(-t \Phi_b(u)) - 1 + t \frac{\Phi_b(u)}{1 + \Phi_b^2(u)} \right\}\,du - t \gamma_b , \]

where
\[ \gamma_b = \int_0^1 \frac{\Phi_b(u)}{1 + \Phi_0^2(u)} \, du - \int_1^\infty \frac{\Phi_0^3(u)}{1 + \Phi_0^2(u)} \, du = \frac{\Phi(b)}{1 + \Phi^2(b)} - \int_1^\infty \frac{\Phi_0^3(u)}{1 + \Phi_0^2(u)} \, du. \]

Using
\[ \exp(v) - 1 \leq v + \frac{v^2}{2} e^v, \ 0 \leq v \leq \gamma, \]
we get
\[ \log E \exp(-t U(b)) \leq \int_0^\infty \frac{-t \Phi_0^3(u)}{1 + \Phi_0^2(u)} \, du + \frac{t^2}{2} e^t f(b) - t \gamma_b \]
\[ = -t \Phi(b) + \frac{t^2}{2} e^t f(b). \]

**Lemma 5.** For all \( t > 0 \) and \( 0 < \lambda \leq 1 \)

\[(17) \quad e^{tz^2} P(-U(z^2) > \lambda z) \to 0 \quad \text{as} \quad z \to \infty. \]

**Proof.** By Lemma 4, for all \( 0 < t < -\lambda / \Phi(z^2) \) with \( 0 < \lambda \leq 1 \) and \( z > 1 \)

\[(18) \quad P(-U(z^2) > \lambda z) \leq \exp(-t \lambda z \Phi(z^2) + \frac{t^2}{2} e^{z^2} f(z^2)). \]

Setting
\[ t = \lambda (e^{z^2} f(z^2) / z^2)^{-1/2} \leq -\lambda / \Phi(z^2) \]
into (18) we obtain
\[ P(-U(z^2) > \lambda z) \leq \exp \left\{ -\lambda z^2 / (e^{z^2} f(z^2))^z + \frac{1}{2} \lambda z^2 - z \lambda \Phi(z^2) / (e^{z^2} f(z^2))^z \right\} \]
\[ \leq \exp \left\{ -\lambda z^2 / (e^{z^2} f(z^2))^z + \frac{1}{2} \lambda z^2 + \lambda \right\}. \]

Since \( f(z^2) \to 0 \) as \( z \to \infty \), we have (17). \[ \square \]

For any \( b > 0 \), set
\[ W_{\rho,k}(b) = \int_b^\infty h_{\rho}(N(s) \vee k, s) \, d\Phi(s), \]
and when \( b > k \) let
\[ \overline{W}_{\rho, k}(b) = \int_k^b h_\rho \left( \text{IN}(s) \lor k, s \right) d \Phi(s). \]

Notice that when \( b > k \) we have
\[
W_{\rho, k}(b) = \int_k^\infty h_\rho \left( \text{IN}(s) \lor k, s \right) d \Phi_b(s) + \int_k^k h_\rho \left( \text{IN}(s) \lor k, k \right)(k/s)^p d \Phi_b(s)
= \int_k^s h_\rho \left( \text{IN}(s), s \right) d \Phi_b(s) + \int_k^k h_\rho(k, s) d \Phi_b(s),
\]
and
\[
(19) \quad V_{\rho, k}^{(1)} = \overline{W}_{\rho, k}(b) + W_{\rho, k}(b).
\]

**Lemma 6.** For every \( \rho \leq 0, k \geq 1 \) and \( \tau > 0 \) there exists an \( m > 0 \) such that for all \( 0 < \lambda \leq 1 \) we have
\[
e^{tz^2} P \left( W_{\rho, k}((mz)^2) < -\lambda mz \right) \rightarrow 0 \quad \text{as} \quad z \to \infty.
\]

**Proof.** First assume \( \rho = 0 \) and \( k = 1 \). In this case the assertion follows from the inequality
\[
W_{0,1}(z^2) = U(z^2) - \Phi_\tau(S_1) \geq U(z^2)
\]
and Lemma 5. For \( \rho < 0 \) and \( k = 1 \), (20) is immediate from the inequality (see (5))
\[
W_{\rho, 1}(z^2) \geq W_{0,1}(z^2).
\]

Now assume \( k > 1 \) and \( \rho \leq 0 \). Notice that if \( S_k < (mz)^2 \) and \( k < (mz)^2 \), then
\[
W_{\rho, k}((mz)^2) = \int_k^\infty h_\rho \left( \text{IN}(s) \lor k, s \right) d \Phi_{(mz)^2}(s)
= \int_1^\infty h_\rho \left( \text{IN}(s) \lor 1, s \right) d \Phi_{(mz)^2}(s) = W_{\rho, 1}((mz)^2).
\]
Thus, for any \( m > 0 \), as soon as \( z \) is large enough so that \( (mz)^2 > k \) we have
\[
P \left( W_{\rho, k}((mz)^2) < -m \lambda z \right) \leq P \left( W_{\rho, 1}((mz)^2) < -m \lambda z \right) + P \left( S_k \geq (mz)^2 \right).
\]
By choosing \( m \) large enough so that
\[
e^{tz^2} P \left( S_k \geq (mz)^2 \right) \rightarrow 0 \quad \text{as} \quad z \to \infty,
\]
we see that the assertion follows. □

Our next goal is to prove the following lemma.

**Lemma 7.** For all \( \rho \leq -\frac{1}{2}, \ k \geq 1 \) and \( 0 < \tau < \infty \)

\[
\lim_{t \to \infty} \inf_{z \to \infty} e^{\tau z^2} P(\overline{W}_{\rho, k}(z^2) < -\lambda z) = 0
\]

for all \( 0 < \lambda \leq 1 \).

**Proof.** We must consider a number of cases separately.

For any \( b > 1 \) and \(-\infty < \rho < \infty\), set

\[
\mu_{\rho}(b) = \int_1^{b} u^{-\rho} d\Phi(u).
\]

**Lemma 7.1.** For all \(-1 \leq \rho < -\frac{1}{2}\)

\[
(21) \quad \mu_{\rho}(b) = o(b^{-\rho - \frac{1}{2}}).
\]

**Proof.** Choose any \( x > 1 \) and \( b > x \), we see that by integrating by parts

\[
\mu_{\rho}(b) = b^{-\rho} \Phi(b) - \Phi(1) + \rho \int_1^{b} u^{-\rho - 1} \Phi(u) \, du
\]

\[
\leq -\Phi(1) + \rho \int_1^{x} u^{-\rho - 1} \Phi(u) \, du - \rho \left[ \int_1^{b} \Phi^2(u) \, du \right]^{\frac{1}{2}} \left[ \int_1^{b} u^{-2\rho - 2} \, du \right]^{\frac{1}{2}}
\]

\[
\leq -\Phi(1) + \rho \int_1^{x} u^{-\rho - 1} \Phi(u) \, du - \frac{\rho}{(-2\rho - 1)^{\frac{1}{2}}} \left[ \int_1^{x} \Phi^2(u) \, du \right]^{\frac{1}{2}} b^{-\rho - \frac{1}{2}}.
\]

Therefore

\[
\limsup_{b \to \infty} \mu_{\rho}(b) b^{\rho + \frac{1}{2}} \leq \frac{-\rho}{(-2\rho - 1)^{\frac{1}{2}}} \left[ \int_1^{x} \Phi^2(u) \, du \right]^{\frac{1}{2}} \text{ for all } x > 1.
\]

Letting \( x \to \infty \) proves (22). □
First we consider

**Case 1.** $\rho < -1$.

In this case $W_{\rho, k}(z^2)$ is bounded below by

$$\mu_{-1}(z^2)/(1+\rho),$$

which by Lemma 7.1 is equal to $o(z)$. This shows that (21) is true with $\lim$ replacing $\lim\inf$.

**Case 2.** $-1 < \rho < -\frac{1}{2}$.

Notice that for $z^2 > 1$

$$W_{\rho, k}(z^2) \geq -\left[\ln(z^2) \vee k\right]^{\rho+1} \mu_{p}(z^2)/(p+1).$$

On the event $B_{z, l} = [\ln(z^2) \leq l z^2]$, with $l > \varepsilon^2$ required below, we have

(23) $-\ln(z^2)^{\rho+1} \mu_{p}(z^2)/(p+1) \geq -l^{\rho+1} \varepsilon^{2p+2} \mu_{p}(z^2)/(p+1)$.

By Lemma 7.1 the right side of (23) is equal to $o(z)$. On the other hand, by Inequality 1

$$P(B_{z, l}) \leq \exp(-z^2 l \log l/2).$$

Since $l$ can be chosen arbitrarily large, we have (21) with lim.

**Case 3.** $\rho = -1$.

For this case we require a lemma.

**Lemma 7.2.**

(24) $\lim_{b \to \infty} (\log b) b^{-1/2} \mu_{-1}(b/(\log b)^2) = 0$

and

(25) $\lim \inf_{b \to \infty} (\log \log b) b^{-1/2} \{\mu_{-1}(b) - \mu_{-1}(b/(\log b)^2)\} = 0.$
PROOF. First, (24) follows from (22). Integrating by parts we see that for \( b > e \) we have

\[
( \log \log b )^{-\alpha} \left\{ \mu_{-1}(b) - \mu_{-1}\left( \frac{b}{(\log b)^2} \right) \right\}
\]

\[
= b^{-\alpha} (\log \log b) \int_{b/(\log b)^2}^b \mu \Phi(u) \text{d}u
\]

\[
\leq - b^{-\alpha} (\log \log b)^{-2} (\log \log b) \Phi\left( \frac{b}{(\log b)^2} \right) - b^{-\alpha} (\log \log b) \int_{b/(\log b)^2}^b \Phi(u) \text{d}u
\]

\[
\leq o(1) (\log \log b) / \log b + \left\{ (\log \log b)^2 \int_{b/(\log b)^2}^b \Phi^2(u) \text{d}u \right\}^{1/2},
\]

using \( z \Phi^2(z) \to 0 \) as \( z \to \infty \) from (1). Hence to prove (25) it suffices to show that

\[
(26) \quad \lim_{b \to \infty} \inf \left( \log \log b \right)^2 \int_{b/(\log b)^2}^b \Phi^2(u) \text{d}u = 0.
\]

Suppose the \( \lim \inf \) in (26) is greater than \( c > 0 \), then for all \( k \geq k_0 \), for some \( k_0 > 0 \), we have

\[
\int_{2^{k+1} - k^2}^{2^{k+1} - ((k+1)^2 \log 2)^2} \Phi^2(u) \text{d}u \geq c / \left( \log \left( (k+1)^2 \log 2 \right) \right)^2 \geq c / \left( \log (k+1) \right)^2.
\]

Since for all large enough \( k \)

\[
2^{(k+1)^2 - k^2} \geq \left( (k+1)^2 \log 2 \right)^2,
\]

this gives for all large \( k \)

\[
\int_{2^{k+1} - k^2}^{2^{k+1} - ((k+1)^2 \log 2)^2} \Phi^2(u) \text{d}u \geq c / \left( \log (k+1) \right)^2,
\]

which contradicts (1). Therefore (26) must be true. \( \square \)

We are now prepared to finish the proof of Case 3.

Choose any \( \tau > 0 \) and \( k \geq 1 \). Notice that on the event

\[
C_{\tau,k} = \left\{ \sup_{s \geq 0} \left\{ \text{IN}(s) - l s \right\} \leq z^2, \quad l \geq k, \right\}
\]

we have

\[
(\text{IN}(s) \vee k) / s \leq l + z^2, \quad \text{when} \quad k \leq s \leq z^2 / (\log(z^2))^2
\]
and

\[(\ln(s) \forall k) / s \leq l + (\log(z^2))^2, \quad \text{when } z^2 / (\log(z^2))^2 \leq s \leq z^2.\]

Setting

\[-z(l) = -\log(l + z^2) \mu_{-1}(z^2 / (\log(z^2))^2) \]

\[-\log(l + (\log(z^2))^2) \left\{ \mu_{-1}(z^2) - \mu_{-1}(z^2 / (\log(z^2))^2) \right\},\]

we see that

\[P(\bar{W}_{-1,k}(z^2) \leq -z(l)) \leq P(\bar{C}_{z,l}),\]

which by Inequality 3 is

\[\leq D(l^{-1}) \exp(-z^2 \log l).\]

Notice by Lemma 7.2 that for all \(l \geq k\) we have

\[\lim_{z \to \infty} \inf \frac{z(l)}{z} = 0,\]

so if we select \(l\) large enough we obtain (21) for \(\rho = -1\).

Finally we consider

**Case 4.** \(\rho = -\frac{1}{2}\).

We shall need a number of lemmas.

**Lemma 7.3.**

\[\lim_{b \to \infty} \frac{\mu_{-\frac{1}{2}}(b)}{(\log b)^{\frac{1}{4}}} = 0.\]

**Proof.** Integrating by parts, we have for \(1 < x < b\),

\[\mu_{-\frac{1}{2}}(b) \leq -\Phi(1) - \frac{1}{2} \int_{1}^{b} u^{-\frac{1}{2}} \Phi(u) \, du \]

\[\leq -\Phi(1) - \frac{1}{2} \int_{1}^{x} u^{-\frac{1}{2}} \Phi(u) \, du + \left[ \int_{x}^{\infty} \Phi^{2}(u) \, du \right] \frac{1}{\sqrt{2}} (\log b)^{\frac{1}{4}}.\]
Therefore for all $x > 1$

$$\limsup_{b \to \infty} \mu_{\nu_x}(b) / (\log b)^{1/2} \leq \left[ \int_1^\infty \Phi^2(u) \, du \right]^{1/2}.$$ 

Letting $x \to \infty$ proves (27). \(\square\)

**Lemma 7.4.**

(28) \(\liminf_{b \to \infty} \log \log b \int_b^{b^2} \Phi^2(u) \, du = 0\).

**Proof.** Suppose that the lim inf in (28) is greater than $c > 0$. Then for all large enough $k$

$$\int_{2^k}^{2^{k+1}} \Phi^2(u) \, du > c / (\log (2^k \log 2)) \geq c/(2k).$$

This contradicts (1). \(\square\)

Set

$$k(z) = \left[ \log z / \log \log z \right] \quad \text{and}$$

$$z_k = z^2 / (\log z)^k, \quad k = 1, \ldots, k(z), \quad z_{k(z)+1} = z.$$ 

Also let

$$S(z) = \sum_{k=1}^{k(z)} \int_{z_{k+1}}^{z_k} \frac{u^{1/2} \, \Phi(u)}{(k \log \log z)^{1/2}} + \int_{z_1}^{z_k} u^{1/2} \, \Phi(u).$$

**Lemma 7.5.**

(29) \(\liminf_{z \to \infty} S(z) = 0\).

**Proof.** Notice that

$$0 \leq S(z) \leq 2^{1/2} \int_1^{z_1} (\log (z^2/u))^{-1/2} u^{1/2} \, \Phi(u) + \int_{z_1}^{z_k} u^{1/2} \, \Phi(u).$$
Integrate this by parts to find that for all large $z$ (use also that (1) implies $\lim_{b \to \infty} b \Phi^2(b) = 0$)

$$S(z) \leq 2^{1/2} (\log \log z)^{-1/2} z^{1/2} \Phi(z_1) - 2^{1/2} (\log z)^{-1/2} z^{1/2} \Phi(z)$$

$$- \int_{1}^{z} (\log (z^{2/\nu}))^{-1/2} u^{-1/2} \Phi(u) \, du + z \Phi(z^2) - z^{1/2} \Phi(z_1)$$

$$- \frac{1}{16} \int_{z_1}^{z^2} u^{-1/2} \Phi(u) \, du$$

$$\leq o(1) + \left[ \int_{z}^{z_1} \Phi^2(u) \, du \right]^{1/2} \left[ \int_{z}^{z_1} (\log (z^{2/\nu}) u)^{-1} \, du \right]^{1/2}$$

$$+ \left[ \int_{z_1}^{z^2} \Phi^2(u) \, du \right]^{1/2} (\log \log z)^{1/2}$$

$$\leq o(1) + \frac{1}{2} \log \log z \left[ \int_{z}^{z_1} \Phi^2(u) \, du \right]^{1/2}.$$

Lemma 7.4 completes the proof. □

**Lemma 7.6.** Let $l > e^2$. Then

(30) \[ P(IN(z^2) > lz^2) < \exp(-lz^2/4) . \]

Also, for all $z$ sufficiently large

(31) \[ P(IN(z) > lz^2/\log z) < \exp(-lz^2/4) , \]

and

(32) \[ P(IN(z_k) > lz^2/(k \log \log z)) < \exp(-lz^2/4) \]

uniformly in $1 \leq k \leq k(z)$.

**Proof.** Obviously (30) follows from Inequality 1. For (31) and (32) we see by the same inequality that for all large $z$

$$P(IN(z) > lz^2/\log z) < \exp \left( -\frac{lz^2}{2 \log z} \log (lz / \log z) \right) ,$$
and uniformly in $1 \leq k \leq k(z)$

$$
P \left( \ln(z_k) > \frac{l z^2}{2} / (k \log \log z) \right) < \exp \left[ - \frac{l z^2}{2} + \frac{z^2 l}{2k} \log (k \log \log z) \right]
$$

$$
\leq \exp \left[ - \frac{l z^2}{2} + \frac{z^2 l}{2} \left( \frac{K}{\log \log z} + \frac{\log \log \log z}{\log \log z} \right) \right],
$$

where

$$
K = \sup_{k \geq 1} \frac{\log k}{k}.
$$

We easily see now that for all $z$ sufficiently large (31) and (32) hold. □

We are now ready to complete the proof of Case 4: $\rho = -\frac{1}{2}$. Choose any $0 < \tau < \infty$ and $l > \varepsilon^2$, and set

$$
E_{z, l} = \left\{ \ln(z^2) \leq l z^2, \ln(z) \leq l z^2 / \log z, \ln(z_k) \leq l z^2 / (k \log \log z) \quad \text{for} \quad 1 \leq k \leq k(z) \right\}.
$$

Notice that

$$
\overline{W}_{1/\varepsilon, k}(z^2) \geq - (\ln(z) \vee k)^{1/\varepsilon} \mu_{1/\varepsilon}(z) - \sum_{k=1}^{k(\varepsilon)} (\ln(z_k) \vee k)^{1/\varepsilon} \int_{z_k}^{z} u^{1/\varepsilon} d \Phi(u)
$$

$$
- (\ln(z^2 \vee k))^{1/\varepsilon} \int_{z_1}^{z^2} u^{1/\varepsilon} d \Phi(u),
$$

which on the event $E_{z, l}$ is for all $z$ sufficiently large

$$
\geq - z^{1/\varepsilon} \left\{ (\log z)^{-1/\varepsilon} \mu_{1/\varepsilon}(z) + S(z) \right\} = - z(l).
$$

Observe that by Lemma 7.6, for all $z$ large

$$
P(\overline{E}_{z, l}) \leq 2 \log z \exp (-l z^2 / 4),
$$

and by Lemmas 7.3 and 7.5 for all $l > 2\varepsilon^2$

$$
\lim_{z \to \infty} \inf_{z(l)} z(l) / z = 0.
$$

Thus by choosing $l$ large enough we obtain (21) for the case $\rho = -\frac{1}{2}$. This completes the proof of Lemma 7. □
Combining Lemmas 6 and 7 with (19) yields

**LEMMA 8.** For all $0 < \tau < \infty$, $\rho \leq -\frac{1}{2}$ and $k \geq 1$ we have

\[
\liminf_{x \to \infty} e^{\tau x^2} P (V_{\rho,k}^{(1)} < -z) = 0.
\]

Next we study the $V_{\rho,k}^{(2)}$ term. Recall (11) above.

**LEMMA 9.** Whenever there exists a $0 < \tau < \infty$ such that

\[
e^{-\tau z^2} \geq P (V_{\rho,k}^{(2)} \leq -z)
\]

for all large enough $z$ with $k \geq 1$ and $\rho \leq 0$, then there exists a constant $0 < b < \infty$ such that

\[
\Phi^2(u) \leq -b \log u \quad \text{for all } u > 0 \text{ small enough.}
\]

**PROOF.** We can assume that $\Phi(u) \to -\infty$ as $u \to 0$, otherwise (35) is trivial. Note

\[
P (V_{\rho,k}^{(2)} \leq -z) = P \left\{ \int_{S_{k+1}}^k h_{\rho}(k \cdot N(s), k) (k/s)^\rho d \Phi(s) < -z \right\}
\]

\[
\geq P (h_{\rho}(k+1, k) (\Phi(k) - \Phi(S_{k+1})) < -z)
\]

\[
= P (-h_{\rho}(k+1, k) \Phi(S_{k+1}) < -z - h_{\rho}(k+1, k) \Phi(k))
\]

which for large enough $z$ is

\[
\geq P (\Phi(S_{k+1}) \leq -\lambda_k z)
\]

where

\[
\lambda_k = -2 / h_{\rho}(k+1, k) > 0.
\]

Set

\[
L(x) = \sup \{s : \Phi(s) \leq x\}, \quad -\infty < x < 0.
\]

Obviously

\[
L(\Phi(x)) \geq s, \quad 0 < s < \infty.
\]

Now
\[ P(\Phi(S_{k+1}) \leq \lambda_k z) = \int_0^{L(-\lambda_k z)} \frac{\mu^k}{k!} e^{-\mu} d\mu \]

\[ \geq \frac{e^{-k}}{(k+1)!} (L(-\lambda_k z))^{k+1} \quad \text{for large enough } z \]

since \( L(x) \downarrow 0 \) as \( x \to -\infty \). From (34) we conclude that there exists a \( 0 < b < \infty \) such that for all large enough \( z \)

\[ e^{-z^2 b} \geq L(-z), \]

which by (36) implies that for all \( u > 0 \) small enough

\[ \exp(-\Phi^2(u)/b) \geq u. \]

This last inequality gives (35). \( \square \)

**Lemma 10.** Suppose (34) holds for some \( \tau > 0, \rho \leq -\frac{1}{2} \) and \( k \geq 1 \). Then

(37) \[ e^{\lambda z^2} P(V_{\rho,k}^{(2)} \leq -z) \to 0 \quad \text{as } z \to \infty \]

for all \( \lambda > 0 \).

**Proof.** Note from definition (11) that

\[ V_{\rho,k}^{(2)} = \int_{\delta_k}^k \delta_{p+k} (k \varphi \ln(k), k) (k/s)^p d\Phi(s) \]

\[ = \int_{\delta_k}^k s^{-\rho} k^{p+1} \zeta_p \left( \frac{k \varphi \ln(k)}{k} \right) d\Phi(s). \]

Integration by parts gives

\[ C_\rho \equiv k^{p+1} \int_0^k s^{-\rho} d\Phi(s) < \infty \]

using (35).

When \( \rho < -1 \) we see that

\[ 0 \geq \zeta_p \left( \frac{k \varphi \ln(k)}{k} \right) \geq 1/(p+1), \]

so that in this case we have...
Thus (37) holds trivially in this case.

Now assume $-1 \leq \rho \leq -\frac{1}{2}$. For $z > 1$, let

$$F_{\rho, z} = \begin{cases} \{N(k) < k z^{1/(\rho+1)}\}, & \text{if } -1 < \rho \leq -\frac{1}{2} \\ \{N(k) < k z^2\}, & \text{if } \rho = -1. \end{cases}$$

On the event $F_{\rho, z}$, $-1 < \rho \leq -\frac{1}{2}$, we have

$$V_{\rho, k}^{(2)} \geq -\frac{C_{\rho}}{\rho + 1} z ;$$

and on the event $F_{-1, z}$ we have

$$V_{-1, k}^{(2)} \geq -C_{-1} z .$$

Applying Inequality 1, it is easy to check that for all $-1 \leq \rho \leq -\frac{1}{2}$ and $0 < \lambda < \infty$,

$$e^{\lambda z^2} P(F_{\rho, z}) \to 0 \quad \text{as} \quad z \to \infty .$$

This completes the proof of (37). □

We are now prepared to finish the proof of Theorem 1 for the case $\rho \leq -\frac{1}{2}$.

Assume that $V_{\rho, k}$ is non-degenerate normal for some $\rho \leq -\frac{1}{2}$ and $k \geq 1$. This implies that there exists a $0 < \tau < \infty$ such that

$$\lim sup \frac{e^{{-z^2}}}{z} \left\{ P(V_{\rho, k} > z) + P(V_{\rho, k} < -z) \right\} < \infty \quad \text{as} \quad z \to \infty .$$

which by Lemma 3 gives

$$\lim sup \frac{e^{{-z^2}}}{z} P(V_{\rho, k}^{(1)} > z) < \infty .$$

Now by (11)

$$P(V_{\rho, k} < -z/2) \geq P(V_{\rho, k}^{(2)} < -z) - P(V_{\rho, k}^{(1)} > z/2) .$$

Therefore from (38) and (39)

$$\lim sup \frac{e^{{-z^2/4}}}{z} P(V_{\rho, k}^{(2)} < -z) < \infty .$$

This in turn implies by Lemma 10 that for all $0 < \gamma < \infty$
we have
\[ P(V_{\rho,k} < -z) \leq P(V_{\rho,k}^{(1)} < -z/2) + P(V_{\rho,k}^{(2)} < -z/2) \]
and from (41) and Lemma 8 we infer that for all \( 0 < \gamma < \infty \)
\[ \lim_{z \to \infty} e^{\gamma z^2} P(V_{\rho,k} < -z) = 0. \]
This contradicts the assumption that \( V_{\rho,k} \) is non-degenerate normal, since if it were
\[ \lim_{z \to \infty} e^{\gamma z^2} P(V_{\rho,k} < -z) = \infty \]
for all \( \gamma \) large enough. This completes the proof of the theorem for \( \rho \leq -\frac{1}{2} \).

For the proof for the case \( \rho > -\frac{1}{2} \), we require a number of additional lemmas. Given \( x, b > 0 \),
\(-\infty < \rho < \infty \) and \( k \geq 1 \) set
\[ U_{\rho,k}(x) = \int_{-x}^{b-x} h_{\rho}(s-x+k,s) d \Phi(s), \]
\[ \overline{U}_{\rho,k}(x) = \int_{x}^{b+\infty} h_{\rho}(s-x+k,s) d \Phi(s), \]
\[ g_{k}(x) = \int_{k}^{x} h_{\rho}(k,s) d \Phi(s), \]
and let \( U_{\rho,k}(x) = \overline{U}_{\rho,k}(x) \). Notice that
\[ V_{\rho,k}(x) = U_{\rho,k}^{b}(x) + \overline{U}_{\rho,k}^{b}(x) + g_{k}(x) = U_{\rho,k}(x) + g_{k}(x). \]
Also note that conditioned on \( \mathbb{N}(b) = m \), \( \overline{U}_{\rho,k}^{b}(x) \) is equal in distribution to \( U_{\rho,k+m}(x+b) \).

**Lemma 11.** For all \( 0 < x, b < \infty, -\infty < \rho, z < \infty \) and \( k \geq 1 \)
\[ P(\overline{U}_{\rho,k}^{b}(x) > z) \leq P(U_{\rho,k}(x+b) > z). \]

**Proof.** Using the above conditional statement for the second equality, we have
\[ P(\overline{U}_{\rho,k}^{b}(x) > z) = \sum_{m=1}^{\infty} P(\overline{U}_{\rho,k}^{b}(x) > z | \mathbb{N}(b) = m) \frac{e^{-b} b^m}{m!} \]
Since \( h_p(u,v) \) being a decreasing function of \( u \) implies \( U_{p,k}(x+b) \geq U_{p,k+m}(x+b), \ m = 1, 2, \ldots \). From this, (43) is obvious. \( \square \)

**Lemma 12.** Whenever there exist \( 0 < 2c < b < \infty \) such that \( \Phi \) is non-constant on \([2c,b] \), then for all \( c/4 \leq x \leq c/2, \ \rho > -\frac{1}{2}, k \geq 1 \) and \( z \geq 1 \) we have

\[
P(U_{\rho,k}(x) \leq B_\rho - zD_\rho) \geq K(c) z^{-1/(2(l+\rho))} \exp \left( -z^{1/(l+\rho)} \log z^{1/(l+\rho)} - \log (ce) \right),
\]

where \( K(c) \) is as in Inequality 2,

\[
B_\rho = (\rho + 1)^{-1} \int_{c/4}^{c/2 + b} s \ d\Phi(s) \quad \text{and} \quad D_\rho = (\rho + 1)^{-1} \int_{3c/2}^{b+c/4} s^{-\rho} d\Phi(s) > 0.
\]

**Proof.** Observe that \([2c,b] \subset [\frac{3c}{2}, b + \frac{c}{4}] \) so that \( D_\rho > 0 \) and

\[
U_{\rho,k}(x) \leq \int_x^{b+x} \frac{s}{\rho + 1} d\Phi(s) - (\ln(c) + k)^{\rho+1} \int_{c+x}^{b+x} \frac{s^{-\rho}}{\rho + 1} d\Phi(s),
\]

which, since \( c/4 \leq x \leq c/2 \), is

\[
\leq \int_{c/4}^{b+c/2} \frac{s}{\rho + 1} d\Phi(s) - (\ln(c))^\rho \int_{3c/2}^{b+c/4} \frac{s^{-\rho}}{(\rho + 1)} d\Phi(s) = B_\rho - \ln(c)^\rho D_\rho.
\]

Now apply Inequality 2. \( \square \)

Our last lemma completes the proof for the case \( \rho > -\frac{1}{2} \).

**Lemma 13.** Whenever there exist \( 0 < 2c < b < \infty \) such that \( \Phi \) is non-constant on \([2c,b] \) then \( V_{p,k} \) is never a non-degenerate normal random variable for any \( \rho > -\frac{1}{2} \) and \( k \geq 1 \).

**Proof.** Choose any \( c/4 \leq x \leq c/2 \) and \( z > 1 \). We have by (42)

\[
P(V_{p,k}(x) \leq -zD_\rho + B_\rho) \geq \ldots
\]
\[
P(U_{\rho,k}^b(x) \leq B_\rho - 2z D_\rho, \overline{U}_{\rho,k}^b(x) + g_k(x) \leq z D_\rho) \geq \]
\[
P(U_{\rho,k}^b(x) \leq B_\rho - 2z D_\rho) - P(\overline{U}_{\rho,k}^b(x) + g_k(x) > z D_\rho)
\]
\[
= P(z) - Q(z).
\]

Note that by Lemma 11, for \(c/4 \leq x \leq c/2\)
\[
Q(z) \leq P(U_{\rho,k}(x + b) + g_k(x + b) \geq z D_\rho - A_\rho) = P(V_{\rho,k}(x + b) \geq z D_\rho - A_\rho),
\]
where
\[
A_\rho = \sup_{c/4 \leq x \leq c/2} \{g_k(x) - g_k(x + b)\} < 0.
\]

Also, by an application of Lemma 12, for all \(1/(1+\rho) < \delta < 2\)
\[
\lim_{z \to \infty} e^{z^\delta} P(z) = \infty.
\]

Thus for all \(c/4 \leq x \leq c/2\) and \(1/(1+\rho) < \delta < 2\) we have
\[
\lim_{z \to \infty} e^{z^\delta} \{P(V_{\rho,k}(x) \leq -z D_\rho + B_\rho) + P(V_{\rho,k}(x + b) \geq z D_\rho - A_\rho)\}
\]
\[
\geq \lim_{z \to \infty} e^{z^\delta} \{P(V_{\rho,k}(x) \leq -z D_\rho + B_\rho) + Q(z)\} \geq \lim_{z \to \infty} e^{z^\delta} P(z) = \infty.
\]

Since (12) gives
\[
P(V_{\rho,k} \leq -z D_\rho + B_\rho) + P(V_{\rho,k} \geq z D_\rho - A_\rho)
\]
\[
\geq \int_{c/4}^{c/2} \{P(V_{\rho,k}(x) \leq -z D_\rho + B_\rho)f_k(x) + P(V_{\rho,k}(x + b) \geq z D_\rho - A_\rho)f_k(x + b)\} \, dx,
\]
we have from the above that for all \(1/(1+\rho) < \delta < 2\)
\[
\lim_{z \to \infty} e^{z^\delta} \{P(V_{\rho,k} \leq -z D_\rho + B_\rho) + P(V_{\rho,k} \geq z D_\rho - A_\rho)\} = \infty.
\]

This shows that \(V_{\rho,k}\) cannot be non-degenerate normal. \(\square\)
References


