A REGULARIZED CONTRAST STATISTIC FOR OBJECT BOUNDARY ESTIMATION: IMPLEMENTATION AND STATISTICAL EVALUATION

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A Regularized Contrast Statistic for Object Boundary Estimation: Implementation and Statistical Evaluation. *

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Abstract

We propose an optimization approach to the estimation a simple closed curve describing the boundary of an object represented in an image. This problem arises in a variety of applications, such as template matching schemes for medical image registration. A regularized optimization formulation with an objective function that measures the normalized image contrast between the inside and outside of a boundary, is considered. Efficient numerical methods are developed to implement the approach and a set of simulation studies are carried out to quantify statistical performance characteristics in the context of boundary determination in emission computed tomography (ECT) images. These results are quite promising. The approach is highly automated which offers some practical advantages over currently used technologies in the medical imaging field.

Keywords: curve parameterization, edge detection, non-linear optimization, simulation.

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1 Introduction

The problem of determining the surface boundary of an object arises frequently in image analysis. In medical imaging, for example, the most widely used approach for registration of brain images from X-ray computerized tomography (CT), magnetic resonance (MR), and emission computed tomography (ECT) is based on correlating head boundaries derived from the different imaging modalities [21]. The extraction of boundary information is a special case of the more general problem of edge detection. There is a substantial literature on general edge detection[18, 25, 24] but methods which incorporate continuity constraints are less well developed [2, 6, 10, 11, 16, 22, 25]. A difficulty with currently available methods for continuous boundary determination is the degree of human intervention necessary to guide the software towards meaningful results, see [2] for some discussion. While there have been considerable advances in the development of graphical interfaces to facilitate this process, in several areas, such as the medical image registration mentioned above, a more automated approach is highly desirable.

Similar to the work of Tan et. al.[25], we attempt to develop an approach to boundary estimation which involves minimal modeling of the image itself. This contrasts with the techniques of Geman et. al.[11] and Chiao[6] for example. The starting point for our method is the notion that object boundaries are characterized as areas where there is rapid change in the gray-scale intensity of the image[11, 25]. Working from this we formulate object boundary curve estimation as a regularized optimization problem, see section 2. To illustrate the approach, we consider the situation where we have 2-dimensional image data, and the target boundary is described by a simple closed curve. There are several applications where this assumption is reasonable[2, 21]. Section 3 describes an efficient numerical algorithm to implement our approach in this setting. The performance of the algorithm is then evaluated on a set of simulated ECT images in section 4. Our results here are used to quantify the relation between reconstruction image quality and the error in boundary estimation algorithm. Section 5 points to some potential extensions of the approach for multichannel image data, texture boundaries, volumetric imaging and multiple objects.
and $z$ is in $\Gamma^0_\theta$ if

$$[x - \theta(s)]\eta_\theta(s_x)' > 0.$$ 

Note that the sets $\Gamma^I_\theta$ and $\Gamma^0_\theta$ form a partition of $\Omega$.

### 2.1 Contrast Statistic

The target boundary will be defined in terms of an ideal perfectly sampled noise-free image. The ideal image $f$ is a function $f : \Omega \rightarrow \mathbb{R}$. We will assume that $f$ is continuously differentiable, i.e. the partial derivatives $\partial_{x_1} f$ and $\partial_{x_2} f$ exist and are continuous for each point $x = (x_1, x_2) \in \Omega$. Since often it is not reasonable to assume that the underlying image is continuously differentiable but only say $L_2$ integrable, we will take our function $f$ to be the underlying image convolved with a spherically symmetric Gaussian kernel whose full-width half maximum (FWHM) is set to be on the order of the resolution of the sampled image data. Note that the Gaussian convolution ensures that the ideal image $f$ is analytic.

Our choice of objective function is based on the notion that one is interested in identifying boundaries corresponding to sharp contrast in the image. We measure the contrast of a boundary $\Gamma_\theta$ by the difference between the average value of $f$ per unit area on the inside and outside of the boundary, i.e.

$$C(\theta) = \left| \frac{I(f)}{I(1)} - \frac{\bar{I}(f)}{\bar{I}(1)} \right|$$

where

$$I(f) = \int_{\Gamma^I_\theta} f(x) dx, \quad I(1) = \int_{\Gamma^I_\theta} dx$$

and

$$\bar{I}(f) = \int_{\Gamma^0_\theta} f(x) dx, \quad \bar{I}(1) = \int_{\Gamma^0_\theta} dx.$$ 

The true boundary will have the property that it maximizes boundary contrast. For our purposes it is more convenient to work with function minimization, the true underlying target boundary of $f$ in $\Omega$ is defined as the simple closed curve $\theta_0$ which minimizes the negative contrast $L(\theta) = -C(\theta)$. Formally

$$\theta_0 = \arg\min_{\theta \in \Theta} L(\theta)$$
Here $W_0^2[0, 1]$ is the second order Sobolev space of periodic functions on $[0, 1]$, see for example [19]. $\Delta$ is a real Hilbert space with inner product

$$<\delta, \phi> = \int_0^1 \delta(s)\phi'(s)ds + \int_0^1 \bar{\delta}(s)\bar{\phi}'(s)ds.$$ 

For any self-adjoint positive semi-definite linear operator $W$ defined on $\Delta$,

$$J(\theta) = <\delta, W\delta>$$

can be used to define a quadratic penalty functional for regularization. Cox and O'Sullivan[7] examine some asymptotic consistency properties of a class of non-linear regularization methods using penalties of this type. In the present paper, we focus on the special case in which

$$J(\delta) = \frac{1}{2} \int_0^1 (\bar{\delta}_1(s)^2 + \bar{\delta}_2(s)^2)ds.$$ 

The sample estimate of the boundary curve is defined as $\theta_{N\lambda} = \delta_{N\lambda} + \tau$ where

$$\delta_{N\lambda} = \text{argmin}_\delta L_{N\lambda}(\delta),$$

the minimization is over all $\delta$ in $\Delta$ for which $\delta + \tau$ is a simple closed curve in $\Omega$.

### 3 Numerical Methods

Our algorithm is based on Newton's method with a trust region step length selection criterion[9]. A variation on the standard implementation is necessary because, as we noted in section 2, the function $L_N$ is not continuous with respect to $\delta$. In order to deal with this, we develop gradients and Hessian formulas for the continuous version, $L$, and from these construct discretized approximations for use in the algorithm. Since $f$ is a smooth function defined on $\Omega$, it is easy to verify that in the class of smooth simple curves the functionals $I(f) = \int_{\Gamma^+} fdx$ and $\bar{I}(f) = \int_{\Gamma^-} fdx$ have continuous second order Frechet derivative at any point $\theta$. The first derivative in the direction of the curve $\xi$ is given by

$$D_\theta I(f, \xi) = \int_0^1 f(\theta(s))[\xi_1(s)\hat{\theta}_2(s) - \xi_2(s)\hat{\theta}_1(s)]ds.$$
reasonable alternatives, however, as we shall see below, the uniform discretization has some nice computational properties. Our basis functions are not differentiable but where ever necessary we will use divided difference approximations for derivatives:

\[ \hat{\theta}(s_j) \approx \frac{\theta(s_{j+1}) - \theta(s_{j-1})}{2h} \quad \text{and} \quad \hat{\theta}(s_j) \approx \frac{\theta(s_{j+1}) - 2\theta(s_j) + \theta(s_{j-1})}{h^2} \]

with \( h = 1/M \). Note that in these expressions we use the periodic extension \( s_0 = s_M \) and \( s_{M+1} = s_1 \). With this

\[ \beta_1 \Sigma \beta'_1 + \beta_2 \Sigma \beta'_2 \approx \beta B_h \beta' \]

where

\[ B_h = \begin{pmatrix} h\Delta_h \Delta_h & 0 \\ 0 & h\Delta_h \Delta_h \end{pmatrix} \]

and \( \Delta_h \) is the \( M \times M \)-Toeplitz matrix representing the second order divided difference

\[
\Delta_h = \frac{1}{h^2} \begin{bmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
1 & 1 & -2
\end{bmatrix}
\]

(All other entries of \( \Delta_h \) are zero.) Let \( \theta \) be the current value of the boundary curve. Approximate gradients and Hessians of \( L_N(\beta) \) are given by

\[ g(\beta) = \partial_\beta L_N^*(\beta) + \lambda B_h \beta \quad \text{and} \quad H(\beta) = \partial^2_\beta L_N^*(\beta) + \lambda B_h. \]

The expressions for \( \partial_\beta L_N^*(\beta) \) and \( \partial^2_\beta L_N^*(\beta) \) are obtained from the continuous version as follows: Suppose \( j \leq M \), then

\[ \partial_\beta L_N^*(\beta)_j \approx D_\theta L(\theta) \xi \]

where \( \xi = (\phi_j, 0) \). The integrals involved in \( D_\theta L(\theta) \) are approximated using the trapezoidal rule and the measured data for \( f \), e.g. for (5)

\[ D_\theta I(\theta) \xi \approx y(\theta(s_j)) \frac{1}{2} \hat{\theta} \Phi(s_j) \]

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\[ c_j = -\frac{1}{2} \partial_{x_1} y(\theta(s_j)) \frac{1}{I_N(1)} + \frac{1}{I_N(1)} \] \[ d_j = \frac{1}{2} \partial_{x_2} y(\theta(s_j)) \frac{1}{I_N(1)} + \frac{1}{I_N(1)} \]

and

\[ e_j = -\frac{1}{2} (\cdot y(\theta(s_j))(\frac{1}{I_N(1)} + \frac{1}{I_N(1)} + \frac{1}{2} \frac{I_N(y)}{I_N(1)^2} + \frac{I_N(y)}{I_N(1)^2}) \]

for \( j = 1, \ldots, M \). Expressions involving partial derivatives of \( y \) are approximated as follows:

Divided differences involving the 2 nearest neighbors are used to construct partial derivatives estimates of \( f \) based on the available grid point measurements, using these values of partials derivatives at \( \theta(s_j) \) are found by linear interpolation of the values at the 4 nearest grid points.

### 3.2 The Newton Iteration

The basic iteration is defined as

\[ \beta^{k+1} = \beta^k - [\alpha_k + H(\beta^k)]^{-1} g(\beta^k) \] (7)

where \( \alpha \) is a trust region step, this step is adjusted so that the resulting curve lies in \( \Omega \). The simplicity of the curve is maintained by ensuring that the lines connecting the points \( \theta^{(k)}(s_j) \) and \( \theta^{(k+1)}(s_j) \) for \( j = 1, 2, \ldots M \) do not intersect each other. Any such intersection leads to a loop in the curve \( \theta^{(k+1)} \), thereby violating simplicity.

The computational complexity of the algorithm is driven by the solution of the linear system in (7). An efficient iterative solution technique is used. Write

\[ H + \alpha I = \begin{bmatrix} \alpha I + D_c + \lambda B_h & E \\ F & \alpha I + D_d + \lambda B_h \end{bmatrix} + R, \]

let \( c = \text{mean} \{c_j\} \), \( d = \text{mean} \{d_j\} \), and \( \tilde{D}_c = D_c - cI \), \( \tilde{D}_d = D_d - dI \), furthermore, let \( H_{11} = (\alpha + c)I + \lambda B_h \) and \( H_{22} = (\alpha + d)I + \lambda B_h \). Using the splitting

\[ H + \alpha I = K - L \]

with \( K = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix} \), the iteration is: \( b^{(k)} = g - R\delta^{(k)} \), and

\[ \delta^{(k+1)}_1 = H_{11}^{-1} (b^{(k)}_1 - E\delta^{(k)}_2 - \tilde{D}_c \delta^{(k)}_1) \]
where
\[ r(t) = \frac{a(1 - e^2)}{1 - e \cos(2\pi t)}. \]

for \( t \in [0, 1] \). We use a one-dimension optimization procedure to find the particular contour which minimizes \( L_N(\theta_a) \) with respect to \( a \), see section 4 for example.

The initial guess \( \tau \) is parameterized so as to have unit speed. The unit speed parameterization guarantees that the sampling of initial curve is more intensive in areas where the local curvature is greatest, see also Hastie and Stuetzle[13]. The unit speed parameterization is

\[ \tau(s) = \theta_a(t(s)) \]  

where, for \( s \in [0, 1] \),
\[ t(s) = \int_0^s \frac{1}{\| \theta_a(u) \|} du. \]

The integrations involved here are computed numerically, thus the resulting curve, \( \tau \), is actually only approximately of unit speed.

3.4 Two Empirical Refinements

Experimentation with the basic Newton algorithm lead to two modifications which were found to improve convergence and accuracy of the estimated boundary curves.

1. Re-focusing \( \Omega \): In general, it is easy to appreciate that the objection function will become flatter and potentially less sensitive to the correct boundary location as the size of \( \Omega \) increases. To address this we use a second application of the Newton algorithm in which the region \( \Omega \) was refocused. The refocused region consists of only those parts of \( \Omega \) which are within \( n \) pixels of the current boundary estimate. The result of this is that during the second application of the Newton algorithm the objective function becomes more sensitive to local changes in image contrast in the neighborhood of the boundary.

2. Re-initialization: This modification was by problems with inappropriate discretization. The accuracy of the boundary curve is affected by the resolution of the discretization which is based on a unit speed parameterization of the initial curve. As a result, if there is a great mismatch between the initial ellipse and the true boundary (see Figure 1 for example),
the right component of the white matter. For each phantom we are interested in estimating the outer boundary between tissue and background.

The phantom data were sub-sampled to obtain images of the type routinely seen in positron emission tomography (PET) scanners $128 \times 128$ pixels with a nominal pixel dimension on the order of 2.0mm. Seven sets of simulated reconstructed emission tomography images were obtained for each of the four phantoms. The reconstruction method was standard filtered backprojection\cite{19} with the reconstruction bandwidth set so that the reconstructed image was as close as possible to the true in a least squares sense. (In practice the reconstruction bandwidth is set by the tomograph operator in order to obtain a 'physiologically reasonable' reconstruction.) Tomograph parameters were set in accordance with the specifications given in Hoffman \textit{et. al.}\cite{14}; 4.5mm detector resolution and a sinogram count array with $128 \times 160$ distance-angle bins. The phantoms were projected into the sinogram domain and pseudo-random measurements were generated according to an inhomogeneous Poisson process with mean determined by the sinogram \cite{23}. Before data generation, the sinograms were rescaled so that the expected count rates for the seven simulated data sets varied linearly on a log$_2$-scale between $10^4$ and $10^8$. This spans the range of counts that appear in actual PET scans.

4.2 Results

We tested the boundary estimation algorithm program on the four groups of data. The initial region $\Omega$ was the entire imaging region and the refocusing technique (see section 3.4) used $n = 10$ pixels. To satisfy the smoothness assumption all data sets were first filtered by convolution with a Gaussian whose FWHM is 2-pixels. The discretization of the boundary curves used $M = 128$ points. For each example, the value of the regularization parameter was set so as to obtain the best visual fit on the true noise-free image. These results were used to define the "true" target boundary curve. Fixing the value of the regularization parameter, the algorithm was then applied to each of the seven simulated ECT reconstructions. The boundary estimation error was evaluated by counting the number
scan the boundary estimation algorithm is made between 15% and 30% more accurate. It is important to realize here that no attempt has been made to optimize the regularization parameter on individual data sets. As a result the potential error characteristic of the approach may in fact be even better than what we see in this experiment.

The average compute time for the boundary curves was 3-5 minutes on a DECstation 5000/200 computer (benchmarked at 27.3 integer MIPS). While no compute time results have been reported for comparable boundary estimation algorithms, we are confident that our algorithm is likely to compare very favorably with the simulated annealing based algorithms of Geman et. al.[11] and Tan et. al.[25]. However, one must appreciate that the goals of these latter methods are also more ambitious.

5 Generalizations

The structure we have developed might be extended to deal with some other situations of interest in practice. We briefly mention three directions for generalization.

5.1 The Contrast Statistic

Multichannel image data, i.e. \( f : \Omega \rightarrow \mathbb{R}^p \) for some \( p > 1 \) arise in a number of applications. For example, the produce Proton density, \( T_1 \)-weighted and \( T_2 \)-weighted images obtained in MR scans. With the multichannel situation the contrast statistic could be extended as

\[
C(\theta) = \sum_{j=1}^{p} w_j C(\theta : f_j)
\]

where \( C(\theta : f_j) \) is the contrast for the \( j \)’th component of the image and \( w_j \) is some weight factor. User specification of the weights could prove difficult. A more sophisticated approach would be to develop a training sample of easily identified pixels on the inside and outside of the object and from the data at these pixels construct a linear discriminant score[12] which optimally separates pixels on the inside from those on the outside of the object. (The idea of training samples has also been mentioned by Geman et. al.[11].) The discriminant score
5.3 Multiple Objects

In many applications it is of interest to resolve multiple objects represented in an image[2, 11, 22, 25]. Our approach would be to attempt to isolate objects one at a time in a recursive segmentation algorithm. This would be similar to the classification and regression tree algorithms discussed in the book by Brieman et al.[5], with our boundary curve algorithm used as a 'splitting rule'. Often the true number of objects would not be known so it would be of interest to construct analogues of cross-validation type methods to determine the number empirically[5]. A complication here, however, is that the stochastic structure of image data (such as reconstructed ECT scans) often involves spatial correlation characteristics which need to be accounted for in applying cross-validation methods. Altman[1] discusses a related issue in the context of bandwidth selection in one-dimensional smoothing problems.

6 Summary

We have studied a new approach for estimating the surface boundary of an object in an image. The method is based on minimization of a regularized functional measuring the normalized contrast in image values between the inside and outside of the boundary. An efficient Newton based algorithm is developed to evaluate the estimator. A set of numerical studies with simulated ECT reconstructions were conducted to quantify the statistical performance characteristics of the approach in the context of ECT scan data. Doubling the count rate in the ECT scan reduces the error in the boundary estimation by between 15% and 30%. These results are quite promising.


Figure Legends

Figure 1

- Layout: As is

- Legend: 1. An Illustration of the Boundary Estimation Algorithm on Simulated ECT Data. (i) the noise free data and the "true" target boundary estimated by algorithm; (ii) a simulated ECT reconstruction based on $N = 0.22 \cdot 10^6$ counts, the initial elliptical curve as well as the optimized solution are shown; (iii) the areas of mismatch between the target boundary and the solution in (ii), the sum of black pixels is used as a measure of error in section 4.2; (iv) the contrast objection function for the elliptical contour (section 3.3), plotted as a function of contour radius.

Figure 2.

- Layout: As is

- Legend: 2. Boundary Estimates for Image Set 1: (i) is the phantom and the target boundary; (ii)-(viii) are boundary estimates for ECT reconstructions based on $10^4$ to $10^8$ expected counts (varying linearly on a log$_2$-scale); (ix) A log-log plot of the error measure as a function of count rate.

Figure 3.

- Layout: As is

- Legend: 3. Boundary Estimates for Image Set 2. (i)-(ix) are as in Figure 2.

Figure 4.

- Layout: As is

- Legend: 4. Boundary Estimates for Image Set 2. (i)-(ix) are as in Figure 2.